Renormalised Chern-Weil forms associated with families of Dirac operators*

Jouko MICKELSSON, Sylvie PAYCHA

February 2, 2008

Abstract

We provide local expressions for Chern-Weil type forms built from superconnections associated with families of Dirac operators previously investigated in [Sc] and later in [PS1].
When the underlying fibration of manifolds is trivial, the even degree forms can be interpreted as renormalised Chern-Weil forms in as far as they coincide with regularised Chern-Weil forms up to residue correction terms. Similarly, a new formula for the curvature of the local fermionic vacuum line bundles is derived using a residue correction term added to the naive curvature formula.
We interpret the odd degree Chern-Weil type forms built from superconnections as Wodzicki residues and establish a transgression formula along the lines of known transgression formulae for η-forms.

Introduction

Chern-Weil formalism in finite dimensions assigns to a connection ∇ on a principal bundle $P \to B$ over a manifold B a form $f(\nabla)$ on B with values in the adjoint bundle Ad P:

$$\begin{array}{rcl} f : \mathcal{C}(P) & \to & \Omega(B, \mathrm{Ad}P) \\ \nabla & \mapsto & f(\nabla), \end{array}$$

which is closed with de Rham cohomology class independent of the choice of connection. Here $\mathcal{C}(P)$ is the space of connections on P and $\Omega(B, W)$ the space of differential forms with values in a vector bundle W over B.

In the context of Ψdo$-$bundles- i.e. bundles with structure group the group of zero order invertible pseudodifferential operators $C\ell_0^*$- the trace used in finite dimensions to build maps $f_j(\nabla) = \mathrm{tr}(\nabla^{2j})$ can be replaced by two [1] natural traces on the algebra $C\ell_0$ of zero order pseudodifferential operators, namely the Wodzikci residue and the leading symbol trace. Such constructions were investigated in [PR] and lead to maps which project down to quotient connections $\bar{\nabla}$ on the quotient bundle $\bar{P}$ with structure group $C\ell_0^*/\left(1 + C\ell_{-\infty}\right)^*$ where $C\ell_{-\infty}$ is the algebra of smoothing operators and $\left(1 + C\ell_{-\infty}\right)^*$ the group of invertibles. In other words, they project down to maps:

$$\begin{array}{rcl} \bar{f} : \mathcal{C}(\bar{P}) & \to & \Omega(B, \mathrm{Ad}\bar{P}) \\ \nabla & \mapsto & \bar{f}(\bar{\nabla}). \end{array}$$

We call such maps *local* in as far as they are insensitive "to smoothing perturbations".

*MSC: 53Z05, 58J40, 58J28, 81T13, 81T50
[1]the only two up to linear combinations [LP]

In contrast, on a principal bundle with structure group $(1+C\ell_{-\infty})^* \subset C\ell_0^*$ one can mimic the ordinary Chern-Weil construction to build Chern classes using the ordinary trace on $C\ell_{-\infty}$. We are concerned in this paper with possible extensions of these Chern forms to Ψdo$-$bundles. Since the ordinary trace on $C\ell_{-\infty}$ extends to linear functionals on $C\ell_0$ obtained from regularised (or weighted) traces, one might want to try to extend the ordinary Chern-Weil constructions to Ψdo$-$bundles using these regularised traces. Such issues were addressed in [PR]; the fact that regularised traces do not yield genuine traces gives rise to obstructions to carrying out the Chern-Weil construction since the regularised Chern forms obtained from regularised traces are not closed. However, it is useful to keep in mind that the obstruction to their closedness can be expressed in terms of local maps in the above sense.

In this paper, we discuss ways to "renormalise" the regularised Chern forms by adding to them local maps in order to turn them into closed forms with de Rham classes independent of the connection. To do so, we compare them with Chern forms previously investigated in [Sc] and later [PS1], which are built from superconnections; in some cases they differ by local expressions so that a renormalisation procedure can indeed be carried out adding local counterterms. More precisely, letting (say in the $\mathbb{Z}_2$-graded case) $\mathbb{A} = D + \nabla$ be a superconnection associated with a Dirac operator D, then the expression

$$\mathrm{tr}^{D^2}(\nabla^{2j}) - \mathrm{tr}^{\,\mathbb{A}^2}(\mathbb{A}^{2j})_{[2j]}$$

-which compares the naive infinite dimensional analog $\mathrm{tr}^{D^2}(\nabla^{2j})$ of the finite dimensional Chern form $\mathrm{tr}(\nabla^{2j})$ and the closed form $\mathrm{tr}^{\,\mathbb{A}^2}(\mathbb{A}^{2j})_{[2j]}$ built from the super connection- is local in the above sense. Here $\mathrm{tr}^{D^2}(B) := \mathrm{fp}_{z=0}\mathrm{TR}(B(D^2+\pi_D)^{-z})$ is the D^2-weighted (or ζ-regularised) trace of B obtained as the finite part at $z=0$ of the meromorphic expansion $\mathrm{TR}(B(D^2+\pi_D)^{-z})$ where B is a form-valued pseudodifferential operator and TR the canonical trace on non integer order pseudodifferential operators [KV]. π_D stands for the orthogonal projection onto the kernel of D.
This "renormalisation" procedure applies to the geometric setup corresponding to families of Dirac operators associated with a trivial fibration of manifolds (see Theorem 1).

In the case of a family of Dirac operators associated with a general fibration of manifolds, such a straightforward "renormalisation procedure" is not possible due to the presence of an extra curvature term arising from a horizontal distribution on the fibration. Indeed, the Chern-Weil forms associated with a superconnection then differs from a weighted Chern form by (a priori) non local terms involving this extra curvature term.
For a family of Dirac operators associated with a general fibration of spin manifolds $\pi : \mathbb{M} \to B$, on the grounds of the family index theorem, we identify Chern forms associated with the superconnection with form components of $\int_{\mathbb{M}/B} \hat{A}(\mathbb{M}/B) \wedge \mathrm{ch}(\mathbb{E}_{\mathbb{M}/B})$ where $\mathbb{E} \to \mathbb{M}$ is a vector bundle over $\mathbb{M}$. The j-th Chern form associated with a superconnection $\mathbb{A}$ introduced in [PS1] (following ideas of [Sc]) has $2j$-form part (see Theorem 2):

$$\mathrm{str}^{\,\mathbb{A}^2}\left(\mathbb{A}^{2j}\right)_{[2j]} = \frac{(-1)^j j!}{(2i\pi)^{\frac{n}{2}}}\left(\int_{\mathbb{M}/B} \hat{A}(\mathbb{M}/B) \wedge \mathrm{ch}(\mathbb{E}_{\mathbb{M}/B})\right)_{[2j]}.$$

On the grounds of the previous discussion, when the fibration is trivial, it differs from renormalised weighted Chern forms by local terms. As it could be expected in analogy with the finite dimensional situation, in the graded case, the first Chern form $\mathrm{str}^{\,\mathbb{A}^2}(\mathbb{A}^2)_{[2]}$ turns out to be proportional to the curvature of the determinant bundle associated with the family of Dirac operators from which the superconnection is built.
But there is also an j-th *residue* Chern form associated with a superconnection $\mathbb{A}$ (which is new to our knowledge) the $2j-1$-th form part of which which reads (see Theorem 2):

$$\mathrm{sres}\left(|\mathbb{A}|^{2j-1}\right)_{[2j-1]} = \sqrt{\pi}\frac{(-1)^j(2j-1)!!}{(2i\pi)^{\frac{n+1}{2}}\,2^{j-1}}\left(\int_{\mathbb{M}/B} \hat{A}(\mathbb{M}/B) \wedge \mathrm{ch}(\mathbb{E}_{\mathbb{M}/B}))\right)_{[2j-1]}.$$

In the non graded case, the second *residue* Chern form (i.e. for $j=2$) turns out to be proportional to the curvature of the gerbe associated with the family of Dirac operators from which the superconnection

is built, which was investigated by Lott [L].
Following a similar scheme to that of Lott [2] we derive a transgression formula for the j-th residue Chern form (see Theorem 3)

$$\operatorname{sres}\left(|\mathbb{A}_\lambda|^{2j-1}\right)_{[2j-1]} = a_j \cdot d\,(\tilde{\eta}_\lambda)_{[2j-2]}\,,$$

using the η-invariant $\tilde{\eta}_\lambda$ (see [BC],[L]) associated with a family of invertible Dirac type operators $D(\lambda) = D - \lambda I$. These perturbed operators differ from that of Lott but match physicists' needs.

The relation to gauge anomalies is explained in the last section of the paper. In particular, a new formula for the curvature of the local fermionic vacuum line bundles (see Theorem 5) is derived using a residue correction term added to the naive curvature formula (see Theorem 4), the latter coming by analogy from the geometry of finite-dimensional Grassmann manifolds, replacing the finite-dimensional trace by a weighted trace.

1 The geometric setup

Let $\pi : E \to M$ be a vector bundle over a closed manifold M. $C\ell_0(M,E)$ denotes the Fréchet Lie algebra of 0-order classical pseudo-differential operators (Ψdo$-$s) acting on smooth sections of E and $C\ell_0^*(M,E)$ the Fréchet Lie group of invertible 0-order classical pseudo-differential operators.
Let $P \to B$ be a $G = C\ell_0^*(M,E)$ principal bundle and $\mathrm{Ad}P = P \times_G C\ell_0(M,E)$ the adjoint bundle, so that locally, $\mathrm{Ad}P_{|_U} \simeq U \times C\ell_0(M,E)$. We equip P with a connection 1-form $\Theta : TP \to C\ell_0(M,E)$ which induces a connection ∇^{Ad} on $\mathrm{Ad}P$. In local coordinates we have $\nabla^{\mathrm{Ad}} = d + [\Theta, \cdot]$ with Θ the above $C\ell_0(M,E)$-valued one form.

The dual bundle $\mathrm{Ad}P^\star$ to $\mathrm{Ad}P$ comes equipped with the dual connection $\left(\nabla^{\mathrm{Ad}}\right)^\star$ defined for any section λ of Ad $P^\star$ and any section σ of $\mathrm{Ad}P$ by

$$d\lambda(\sigma) = \left(\left(\nabla^{\mathrm{Ad}}\right)^\star \lambda\right)(\sigma) + \lambda(\nabla^{\mathrm{Ad}}\sigma).$$

On the other hand, G acts on the space $C^\infty(M,E)$ of smooth sections of E and the associated vector bundle $\mathcal{E} = P \times_G C^\infty(M,E)$ comes equipped with the connection ∇, locally of the form $\nabla = d + \Theta$. Then, locally $\nabla^{\mathrm{Ad}} = d + [\Theta, \cdot]$ and $\left(\nabla^{\mathrm{Ad}}\right)^* = d - [\Theta, \cdot]$. It is therefore convenient to write $\nabla^{\mathrm{Ad}}\sigma = [\nabla, \sigma]$ for any section σ of $\mathrm{Ad}P$ and $\left(\nabla^{\mathrm{Ad}}\right)^* \lambda = [\nabla, \lambda]$ for any section λ of $\mathrm{Ad}P^\star$. With these notations we have:

$$d\,(\lambda(\sigma)) = [\nabla, \lambda](\sigma) + \lambda([\nabla, \sigma]).$$

The group $(1 + C\ell_{-\infty}(M,E))^*$, where $C\ell_{-\infty}(M,E)$ denotes the algebra of smoothing operators, is a normal subgroup of $C\ell_0^*(M,E)$. Quotienting $C\ell_0^*(M,E)$ by $(1 + C\ell_{-\infty}(M,E))^*$ yields quotient bundles $\bar{P} \to B$ and $\bar{\mathcal{E}} = \bar{P} \times_{\bar{G}} C^\infty(M,E)$ with structure group

$$\bar{G} := C\ell_0^*(M,E)/\left(1 + C\ell_{-\infty}(M,E)\right)^*$$

equipped with the induced connection $\bar{\nabla}$.
Let $\mathcal{C}(P)$ and $\mathcal{C}(\bar{P})$ denote the space of connections on P and $\bar{P}$.

Definition 1 *We call a map*

$$\begin{array}{rcl} f : \mathcal{C}(P) & \to & \Omega(B, C\ell(P)) \\ \nabla & \mapsto & f(\nabla) \end{array}$$

[2] We derive a complete proof clarifying some steps in Lott's proof. Our proof is carried out for operators $D(\lambda) = D - \lambda\, I$ (which are differential operators) but it easily extands to operators $D_\alpha = D + h_\alpha(D)$ (which are pseudo-differential operators) used by Lott where h_α is a smooth function with compact support.

local *whenever it projects down to:*

$$\begin{array}{rcl}\bar f:\mathcal{C}(\bar P) & \to & \Omega(B, C\ell(\bar P))\\ \nabla & \mapsto & \bar f(\bar\nabla).\end{array}$$

Let $C\ell(\mathcal{E}) = P\times_G C\ell(M,E)$ denote the bundle of classical pseudo-differential operators with fibre the whole algebra $C\ell(M,E)$ of classical pseudo-differential operators acting on sections of E. Clearly, $\mathrm{Ad}P \subset C\ell(\mathcal{E})$ is a subbundle of $C\ell(\mathcal{E})$.

In view of the following constructions, it is useful to mention that when $M=\{*\}$ is a point, then $E=V$ is a vector space, $C\ell(M,E)=C\ell_0(M,E)=\mathrm{End}(V)$, $C\ell^*(M,E)=C\ell_0^*(M,E)=\mathrm{GL}(V)$ so that $P\to B$ boils down to an ordinary $GL(V)$-principal bundle and both $C\ell(\mathcal{E})\to B$ and $\mathrm{Ad}P\to B$ boil down to its adjoint bundle $\mathrm{Ad}P = P\times_G \mathrm{End}(V)$ for the adjoint action of $GL(V)$ on $\mathrm{End}(V)$. Thus, Ψdo$-$bundles can be seen as natural generalisations of ordinary principal bundles.

2 Q-weighted traces (a short review)

A first atempt to generalise to Ψdo$-$bundles the construction of Chern-Weil forms on ordinary bundles, is to use regularised (or weighted) traces of powers of the curvature as an Ersatz for ordinary traces of powers of the curvature which provide representatives of Chern-Weil classes in finite dimensions [PR]. We give a brief review of weighted traces of classical pseudo-differential operators.

Let $Q\in C\ell(M,E)$ be an invertible admissible elliptic operator of positive order q, where by admissible we mean that its leading symbol admits a spectral cut θ [3]. If Q is not invertible, we replace it by $Q+\pi_Q$ where π_Q is the orthogonal projection onto the kernel of Q.

An invertible admissible elliptic operator Q has complex powers

$$Q_\theta^z = \frac{1}{2i\pi}\int_{\Gamma_\theta} \lambda^z (Q-\lambda)^{-1}\, d\lambda$$

where Γ_θ is a contour around the spectral cut and hence its logarithm $\log_\theta Q = \frac{d}{dz} Q^z|_{z=0}$ which is not classical anymore. In applications to follow, Q is non negative self-adjoint so that $\theta=\pi$ can be chosen as a spectral cut. We shall henceforth drop out the explicit mention of the spectral cut writing simply Q^{-z} and $\log Q$. Recall that for any $A\in C\ell(M,E)$ and provided Q has positive order, the map $z\mapsto \mathrm{TR}\,(A\,Q^{-z})$ is meromorphic with simple pole at 0 and its finite part at 0

$$\mathrm{tr}^Q(A) := \mathrm{fp}_{z\to 0}\mathrm{TR}\left(A\,Q^{-z}\right)$$

is called the Q-weighted (or ζ-regularised) trace of A. Here TR is the canonical trace on non integer order classical Ψdo$-$s [KV]. Even though it is not cyclic and hence not a genuine trace, the Q-weighted trace deserves the name of a trace in as far as it coincides with the ordinary trace on trace-class operators and hence on $C\ell_{-\infty}(M,E)$ and therefore extends it to a linear map on $C\ell(M,E)$.

In contrast, the Wodzicki residue defined for $A\in C\ell(M,E)$ by

$$\mathrm{res}(A) = \frac{1}{(2\pi)^n}\int_{S^*M} \mathrm{tr}_x\left(\sigma_A(x,\xi)\right)_{-n}\, dx\, d_S\xi$$

vanishes on trace-class operators and hence on $C\ell_{-\infty}(M,E)$. Here S^*M stands for the cotangent unit sphere, $d_S\xi$ the canonical volume measure on SM, σ_A is the symbol of A, tr_x the fibrewise trace and the subscript $-n$ stands for the $-n$ (positively) homogeneous part of the symbol.

When A is a differential operator we have [PS2]:

$$\mathrm{tr}^Q(A) = -\frac{1}{q}\mathrm{res}\,(A\,\log Q) \tag{1}$$

[3] An operator $Q\in C\ell(M,E)$ of positive order is called *admissible* if there is a proper subsector of $\mathbb{C}$ with vertex 0 which contains the spectrum of the leading symbol $\sigma_L(Q)$ of Q. Then there is a half line $L_\theta = \{r\,e^{i\theta}, r>0\}$ (a spectral cut) with vertex 0 and determined by an Agmon angle θ which does not intersect the spectrum of Q. If Q is invertible, then $\bar L_\theta = \{r\,e^{i\theta}, r\geq 0\}$ does not intersect the spectrum of Q.

where the residue on the r.h.s is defined by the above formula in spite of $A\log Q$ not being classical anymore. The fact that A is differential ensures that the residue is well-defined.
In general $\mathrm{tr}^Q(A)$ depends on Q for a given $A\in C\ell(M,E)$; given two weights $Q_1,Q_2\in C\ell(M,E)$ with positive orders q_1,q_2 and same spectral cut we have:

$$\mathrm{tr}^{Q_1}(A)-\mathrm{tr}^{Q_2}(A)=-\mathrm{res}\left(A\left(\frac{\log Q_1}{q_1}-\frac{\log Q_2}{q_2}\right)\right). \tag{2}$$

Also, tr^Q is not cyclic: the obstruction to the cyclicity of tr^Q is measured by a Wodzicki residue:

$$\mathrm{tr}^Q([A,B])=-\frac{1}{q}\mathrm{res}\left(A\,[B,\log Q]\right), \tag{3}$$

where now the residue is applied to a genuine classical operator since the bracket $[B,\log Q]$ is classical.

We shall need the following technical lemma (see [BGV] Lemma 9.35).

Lemma 1 *Let f be a smooth function on $]0,+\infty[$ with asymptotic behaviour at 0 of the type*

$$f(\varepsilon)\sim_{\varepsilon\to 0}\sum_{j=0}^{\infty}a_j\varepsilon^{\alpha-j}$$

for some real number α (depending on f) and such that for large enough ε,

$$|f(\varepsilon)|\le Ce^{-\varepsilon\,\lambda}$$

for some $\lambda>0$, $C>0$. Then its Mellin transform

$$z\mapsto\mathcal{M}(f)(z):=\frac{1}{\Gamma(z)}\int_0^\infty\varepsilon^{z-1}f(\varepsilon)\,dt$$

defines a meromorphic map on the complex plane (which turns ut to be holomorphic at $z=0$) and

$$\mathrm{fp}_{\varepsilon=0}f(\varepsilon)=\mathrm{fp}_{z=0}\mathcal{M}(f)(z)=\mathcal{M}(f)(0).$$

In particular, if $f(\varepsilon)=\sqrt{\varepsilon}\,g(\varepsilon)$ then

$$\mathrm{fp}_{\varepsilon=0}f(\varepsilon)=\sqrt{\pi}\,\mathrm{res}_{z=0}\left(\mathcal{M}(g)(z+\frac{1}{2})\right).$$

Proof: The first part of the lemma is well known (see e.g. [BGV] Lemma 9.35). Let us check the formula relating finite parts of $f(\varepsilon)=\sqrt{\varepsilon}g(\varepsilon)$ and its Mellin transform.

$$\begin{aligned}\mathrm{fp}_{\varepsilon=0}f(\varepsilon)&=\mathrm{fp}_{z=0}\mathcal{M}(f)(z)\\&=\mathrm{fp}_{z=0}\frac{1}{\Gamma(z)}\int_0^\infty\sqrt{\varepsilon}\,\varepsilon^{z-1}g(\varepsilon)d\varepsilon\\&=\mathrm{fp}_{z=0}\frac{1}{\Gamma(z)}\int_0^\infty\varepsilon^{z+\frac{1}{2}-1}g(\varepsilon)d\varepsilon\\&=\mathrm{fp}_{z=0}\frac{\Gamma(z+\frac{1}{2})}{\Gamma(z)}\mathcal{M}(g)(z+\frac{1}{2})\\&=\Gamma(\frac{1}{2})\,\mathrm{fp}_{z=0}\left(z\,\mathcal{M}(g)(z+\frac{1}{2})\right)\\&=\sqrt{\pi}\,\mathrm{res}_{z=0}\left(\mathcal{M}(g)(z+\frac{1}{2})\right).\end{aligned}$$

□

The Mellin transform provides a stepping stone between heat-kernel regularisation and ζ-regularisation methods:

Proposition 1 *For any $Q \in C\ell(M,E)$ non negative self-adjoint elliptic and any $A \in C\ell(M,E)$ with vanishing Wodzicki residue:*

$$\mathrm{tr}^Q(A) = \mathrm{fp}_{\varepsilon=0}\mathrm{tr}\left(A\, e^{-\varepsilon Q}\right).$$

Proof: This follows from Lemma 1 applied to $f(\varepsilon) = \mathrm{tr}(A\, e^{-\varepsilon Q})$. □

3 Q-weighted Chern forms

We define weighted Chern-Weil forms as in [PR] and briefly recall the obstructions to the closedness. Weighted traces extend to Ψdo$-$valued forms in the following manner. Given a Ψdo$-$vector bundle $\mathcal{E}$, Q is a section of $C\ell(\mathcal{E})$ which is elliptic, admissible and has positive constant order q. Note that these properties, ellipticity, admissibility and of constant order q are invariant under the adjoint action of the group $C\ell^*(M,E)$ of invertible classical pseudo-differential operators. The definition of the Q-weighted trace and the Wodzicki residue then extend to Ψdo$-$valued forms setting for $b \in U \subset B$ and $\alpha \otimes A \in \Omega\left(U, C\ell(\mathcal{E})\right)$, with $\alpha \in \Omega(U)$, $A \in C^\infty(U, C\ell(\mathcal{E}))$:

$$\mathrm{tr}^Q(\alpha \otimes A)(b) := \alpha(b) \otimes \mathrm{tr}^{Q_b}(A(b)); \quad \mathrm{res}(\alpha \otimes A)(b) := \alpha(b) \otimes \mathrm{res}(A(b)).$$

Properties (1), (2) (3) extend in a straighforward manner to forms:

$$\begin{aligned} \mathrm{tr}^Q(\alpha) &= -\frac{1}{q}\,\mathrm{res}(\alpha\, \log Q) \\ \mathrm{tr}^{Q_1}(\alpha) - \mathrm{tr}^{Q_2}(\alpha) &= -\,\mathrm{res}\left(\alpha\left(\frac{\log Q_1}{q_1} - \frac{\log Q_2}{q_2}\right)\right) \\ \mathrm{tr}^Q([\alpha,\beta]) &= -\frac{1}{q}\,\mathrm{res}(\alpha\,[\beta, \log Q]), \end{aligned} \tag{4}$$

where the first identity holds whenever α is a differential operator valued form whereas the others hold for any $C\ell\left(\mathcal{E}\right)$-valued forms α, β.
The Wodzicki residue commutes with differentiation $[\nabla, \mathrm{res}] = 0$ whereas weighted traces do not. The obstruction is measured in terms of a Wodzicki residue. Indeed, it follows from (4) that locally, $[d\,\mathrm{tr}^Q](\alpha) = -\frac{1}{q}\mathrm{res}(\alpha\, d\log Q)$ as a result of which, writing $\nabla^{\mathrm{Ad}} = d + [\theta, \cdot]$ in local coordinates, we have:

$$\begin{aligned} [\nabla, \mathrm{tr}^Q](\alpha) &= d\left(\mathrm{tr}^Q(\alpha)\right) - \mathrm{tr}^Q([\nabla, \alpha]) \\ &= d\left(\mathrm{tr}^Q(\alpha)\right) - \mathrm{tr}^Q(d\,\alpha) - \mathrm{tr}^Q([\theta, \alpha]) \\ &= [d\,\mathrm{tr}^Q](\alpha) - \frac{1}{q}\,\mathrm{res}\left(\alpha\,[\theta, \log Q]\right) \\ &= -\frac{1}{q}\mathrm{res}(\alpha\, d\log Q) - \frac{1}{q}\,\mathrm{res}\left(\alpha\,[\theta, \log Q]\right) \\ &= -\frac{1}{q}\mathrm{res}\left(\alpha\,[\nabla, \log Q]\right) \quad \forall \alpha \in \Omega\left(B, C\ell(\mathcal{E})\right). \end{aligned} \tag{5}$$

The curvature $\Omega = \left(\nabla^{\mathrm{Ad}}\right)^2$ of ∇^{Ad} lies in $\Omega^2\left(P, C\ell\left(\mathcal{E}\right)\right)$ so that the Q-weighted trace $\mathrm{tr}^Q(\Omega^i)$ defines a $2i$-form on B. The following proposition tells us that the obstruction to the closedness is local in the sense of the above definition.

Proposition 2 *The exterior differential of the weighted Chern-Weil form $\mathrm{tr}^Q(\Omega^i)$ is local, i.e. of the form $d\mathrm{tr}^Q(\Omega^i) = f_i(\bar{\nabla})$ for some $\bar{f}_i : \mathcal{C}(\bar{P}) \to \Omega(C\ell(\bar{P}))$.*

Proof: Since $\nabla^{\mathrm{Ad}}(\Omega) = [\nabla, \Omega^i] = 0$, by (5) we have

$$\begin{aligned} d\mathrm{tr}^Q(\Omega^i) &= [\nabla, \mathrm{tr}^Q](\Omega^i) + \mathrm{tr}^Q([\nabla, \Omega^i]) \\ &= [\nabla, \mathrm{tr}^Q](\Omega^i) + \sum_{j=1}^{i} \mathrm{tr}^Q(\Omega^j\,[\nabla, \Omega]\,\Omega^{i-j}) \\ &= [\nabla, \mathrm{tr}^Q](\Omega^i) \\ &= -\frac{1}{q}\mathrm{res}\left(\Omega^i\,[\nabla, \log Q]\right). \end{aligned}$$

Since the Wodzicki residue vanishes on smoothing operators,

$$d\,\mathrm{tr}^Q(\Omega^i) = d\,\mathrm{tr}^Q(\bar{\Omega}^i) = \bar{f}(\bar{\nabla}),$$

and hence the locality property of the obstruction to the closedness. $\square$.

4 From superconnections to Chern-Weil type forms

We review and extend constructions of Chern-Weil type forms carried out in [PS1] using superconnections. Let $\mathcal{E}$ be a vector bundle associated with a Ψdo$-$principal bundle P as before.

4.1 Chern forms associated with superconnections

- **The $\mathbb{Z}_2$-graded case:** Let us assume that $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ is a $\mathbb{Z}_2$-graded super bundle on B. The canonical trace TR for non integer order operators in $C\ell(\mathcal{E})$ is replaced by the super canonical trace sTR whereas weighted traces tr^Q for operators in $C\ell(\mathcal{E})$ are replaced by weighted supertraces str^Q with respect to even weights $Q = Q^+ \oplus Q^-$. They vanish on odd Ψdos$-$and give the difference of weighted traces on even Ψdos$-$:

$$\mathrm{sTR}(A) := \mathrm{TR}(A^{++}) - \mathrm{TR}(A^{--}); \quad \mathrm{str}^Q(A) := \mathrm{tr}^{Q^+}(A^{++}) - \mathrm{tr}^{Q^-}(A^{--})$$

with obvious notations.
This grading combined with the $\mathbb{Z}_2$-grading on forms $\Omega(B,\mathcal{E}) = \Omega^{ev}(B,\mathcal{E}) \oplus \Omega^{od}(B,\mathcal{E})$ gives rise to:

$$\Omega^+(B,\mathcal{E}) = \Omega^{ev}(B,\mathcal{E}^+) \oplus \Omega^{od}(B,\mathcal{E}^-); \quad \Omega^-(B,\mathcal{E}) = \Omega^{od}(B,\mathcal{E}^+) \oplus \Omega^{ev}(B,\mathcal{E}^-).$$

Definition 2 *[BGV] A superconnection is an odd-parity first order differential operator*

$$\mathbb{A} : \Omega^{\pm}(B,\mathcal{E}) \to \Omega^{\mp}(B,\mathcal{E})$$

which satisfies the (graded) Leibniz rule. If $\alpha \in \Omega(U)$ for some open subset $U \subset B$ and $B \in C^\infty(U, C\ell(\mathcal{E}))$ then

$$\mathbb{A}(\alpha \otimes B) = d\alpha \otimes B + (-1)^{|\alpha|}\alpha \otimes \mathbb{A}\,B.$$

The curvature $\mathbb{A}^2$ of a superconnection $\mathbb{A}$ on $\mathcal{E}$ lies in $\Omega(B, C\ell(\mathcal{E}))$. Following Quillen [Q] we say a superconnection $\mathbb{A}$ on $\mathcal{E}$ is associated with a smooth family of elliptic differential operators $\{D_b, b \in B\}$ whenever $\mathbb{A}_{[0]} = D$.

Example 1 $\mathbb{A} = D + \nabla$ *defines a particular super connection associated with D with curvature*

$$\mathbb{A}^2 = D^2 + \nabla^{\mathrm{Ad}} D + \Omega = Q + [\nabla, D] + \Omega$$

where we have set $Q = D^2$. Here $[\nabla, D] = \nabla D + D\nabla$ is the anticommutator.

Remark 1 *In the following we systematically use graded commutators of operator valued forms: anticommutator for odd-odd forms and usual commutator otherwise.*

- **The non graded case:** Let us assume that $\mathcal{E}$ is an ordinary vector bundle on B. Following Quillen, we introduce an extra grading σ such that $\sigma^2 = 1$ and build the right $\mathbb{C} \oplus \mathbb{C}\,\sigma$-module:

$$C\ell_\sigma(\mathcal{E}) := C\ell(\mathcal{E}) \bigotimes (\mathbb{C} \oplus \mathbb{C}\,\sigma).$$

Odd degree Ψdos$-$lie in $C\ell(\mathcal{E}) \bigotimes (\mathbb{C}\,\sigma)$ whereas $C\ell(\mathcal{E})$ is identifued with even degree Ψdos$-$. The ordinary canonical trace TR extends to non integer order operators in $C\ell_\sigma(\mathcal{E})$ by $\mathrm{sTR}(\alpha + \sigma\beta) = \mathrm{TR}(\beta)$ so that weighted traces tr^Q are replaced by

$$\mathrm{str}^Q(\alpha + \sigma\beta) := \mathrm{tr}^Q(\beta).$$

These definitions extend to the space $\Omega_\sigma(B,\mathcal{E})$ of $C\ell_\sigma(\mathcal{E})$-valued forms on B in a straightforward manner.

Definition 3 *[BGV] A superconnection is a first order differential operator*

$$\mathbb{A} : \Omega_\sigma(B, \mathcal{E}) \to \Omega_\sigma(B, \mathcal{E})$$

which commutes with σ and satisfies the (graded) Leibniz rule.

As in the even case, it is associated with a family $\{D_b, b \in B\}$ of elliptic differential operators whenever $\mathbb{A}_{[0]} = D$.

Example 2 $\mathbb{A} := \sigma D + \nabla$ *is a particular superconnection associated with D, the curvature of which reads* [4]

$$\mathbb{A}^2 = D^2 + \nabla^{\mathrm{Ad}}(\sigma D) + \Omega = Q + [\nabla, \sigma D] + \Omega$$

with $Q = (\sigma D)^2 = D^2$ as before.

Let us recall from [Sc] (see also [PS1]) that weighted traces can be extended to include weights $\mathbb{A}^2$ which are Ψdo$-$valued forms and analogs of Chern-forms can be constructed, which turn out to be closed. Writing

$$\mathbb{A}^2 = \mathbb{A}^2_{[0]} + \mathbb{A}^2_{[1]} + \mathbb{A}^2_{[2]} = D^2 + \mathbb{A}^2_{[>0]}$$

where the subscript $[j]$ stands for the j-th degree part, and $[> 0]$ for non no zero degree part, can be useful to derive explicit expansions in increasing form degree. For example,

$$\begin{aligned}
(\lambda - \mathbb{A}^2)^{-1} &= \left(\lambda - D^2 - \mathbb{A}^2_{[>0]}\right)^{-1} \\
&= \sum_{j=0}^{K} \left(\lambda - D^2\right)^{-1} \mathbb{A}^2_{[>0]} \left(\lambda - D^2\right)^{-1} \cdots \mathbb{A}^2_{[>0]} \left(\lambda - D^2\right)^{-1} \\
&+ S_K(D^2, \mathbb{A}^2_{[>0]}, \lambda), \qquad (6)
\end{aligned}$$

where S_K has form degree $> K$ and where $\mathbb{A}^2_{[>0]} \left(\lambda - D^2)\right)^{-1}$ arises j times in the j-th term of the sum. By convention the $j = 0$ term reduces to $\left(\lambda - D^2)\right)^{-1}$. Hence, for any positive integer K

$$\left((\lambda - \mathbb{A}^2)^{-1}\right)_{[K]} = \sum_{j=0}^{K} \left(\lambda - D^2\right)^{-1} \mathbb{A}^2_{[>0]} \left(\lambda - D^2)\right)^{-1} \cdots \mathbb{A}^2_{[>0]} \left(\lambda - D^2\right)^{-1}$$

has a finite expansion in increasing form degree. Also, whenever $\mathbb{A}_{[0]} = D$ is invertible, so is $\mathbb{A}^2$ invertible and its modulus $|\mathbb{A}| := \left(\mathbb{A}^2\right)^{\frac{1}{2}}$ can be defined using a contour integration (cfr Section 2):

$$|\mathbb{A}| = \frac{i}{2\pi} \int_\Gamma \sqrt{\lambda}\, (\mathbb{A}^2 - \lambda)^{-1}\, d\lambda$$

where Γ is a contour around the spectrum of D^2 which is a subset of $\mathbb{R}^+$.
In general, we set

$$|\mathbb{A}| := \sqrt{\mathbb{A}^2 + \pi_{\mathbb{A}}}$$

where $\pi_{\mathbb{A}}$ is the orthogonal projection onto the kernel of $\mathbb{A}^2_{[0]}$. This defines a form provided Ker $\mathbb{A}^2_{[0]}$ has constant dimension.

Remark 2 *Note that for any $\alpha \in \Omega(B, C\ell(\mathcal{E}))$,*

$$\mathrm{str}^{\mathbb{A}^2}(\alpha) = \mathrm{fp}_{z=0}\mathrm{sTR}\left(\alpha\, \left(\mathbb{A}^2 + \pi_{\mathbb{A}}\right)^{-z}\right) = \mathrm{fp}_{\varepsilon=0}\mathrm{str}\left(\alpha\, e^{-\varepsilon\, \mathbb{A}^2}\right)$$

since $\mathbb{A}^2$ is a differential operator-valued form and hence has vanishing Wodzicki residue.

The following proposition extends results of [PS1].

[4] Here D commutes with σ whereas ∇ anticommutes with σ.

Proposition 3 *Let P be a polynomial function. Forms* $\mathrm{str}^{\mathbb{A}^2}(P(\mathbb{A}^2))$ *and* $\mathrm{sres}\,(P(|\mathbb{A})|)$ *associated with a superconnection* $\mathbb{A}$ *are closed. Their de Rham class is independent of the choice of connection one forms* $\mathbb{A}_{[1]}$.

Remark 3 *The residue form* $\mathrm{sres}\,(P(|\mathbb{A}|))$ *is in fact insensitive to the projection* $\pi_{\mathbb{A}}$ *which is a smoothing operator and hence does not affect the Wodzicki residue.*

Proof: We extend the argument used in [PS1] for the closedness of forms $\mathrm{str}^{\mathbb{A}^2}(\mathbb{A}^{2i})$ to any $\mathrm{str}^{\mathbb{A}^2}(P(\mathbb{A}))$. Equations (5) and (3) extend replacing the connection ∇ by the superconnection $\mathbb{A}$ and the weight Q by the Ψdo$-$valued form $\mathbb{A}^2$ of order 2 [PS1] and we have

$$\begin{aligned} d\,\mathrm{str}^{\mathbb{A}^2}\left(P(\mathbb{A}^2)\right) &= [\mathbb{A}, \mathrm{str}^{\mathbb{A}^2}]\left(P(\mathbb{A}^2)\right) + \mathrm{str}^{\mathbb{A}^2}\left([\mathbb{A}, P(\mathbb{A}^2)]\right) \\ &= -\frac{1}{2}\mathrm{sres}\left(P(\mathbb{A}^2)[\mathbb{A}, \log(\mathbb{A}^2+\pi_{\mathbb{A}})]\right) \\ &= -\frac{1}{2}\frac{d}{dt}_{t=0}\left(\frac{i}{2\pi}\int_\Gamma \lambda^t\,\mathrm{sres}\left(P(\mathbb{A}^2)[\mathbb{A}, (\mathbb{A}^2+\pi_A-\lambda)^{-1}]\right)d\lambda\right) \\ &= 0 \end{aligned}$$

since $[\mathbb{A}, (\mathbb{A}^2+\pi_A-\lambda)^{-1}]$ is smoothing.

Similarly,

$$d\,\mathrm{sres}\,(P(|\mathbb{A}|)) = \mathrm{sres}\,([\mathbb{A}, P(|\mathbb{A}|)]) = 0$$

since $[\mathbb{A}, \log P(|\mathbb{A}|)] = 0$.
The forms are therefore closed.
Let us check that their de Rham classes are independent of the choice of superconnection. Let $\mathbb{A}_t$ be a smooth one parameter family of superconnections then for any monomial $P(\mathbb{A}^2) = \mathbb{A}^{2i}$

$$\begin{aligned} &\frac{d}{dt}\left(\mathrm{str}^{\mathbb{A}_t^2}\left(\mathbb{A}_t^{2i}\right)\right) \\ =\; &\frac{d}{dt}\left(\mathrm{fp}_{\varepsilon=0}\mathrm{str}\left(\mathbb{A}_t^{2i}\,e^{-\varepsilon\,\mathbb{A}_t^2}\right)\right) \\ =\; &\sum_{j=1}^{i}\mathrm{fp}_{\varepsilon=0}\left(\mathrm{str}\left(\mathbb{A}_t^{2(i-j-1)}\,[\mathbb{A}_t, \dot{\mathbb{A}}_t]\,\mathbb{A}_t^{2j}\,e^{-\varepsilon\,\mathbb{A}_t^2}\right) - \varepsilon\,\mathrm{str}\left(\mathbb{A}_t^{2i}\,[\mathbb{A}_t, \dot{\mathbb{A}}_t]\,e^{-\varepsilon\,\mathbb{A}_t^2}\right)\right) \\ =\; &i\,\mathrm{fp}_{\varepsilon=0}\left(\mathrm{str}\left([\mathbb{A}_t, \mathbb{A}_t^{2(i-1)}\,\dot{\mathbb{A}}_t\,e^{-\varepsilon\,\mathbb{A}_t^2}]\right) - \varepsilon\,\mathrm{str}\left([\mathbb{A}_t, \mathbb{A}_t^{2i}\,\dot{\mathbb{A}}_t\,e^{-\varepsilon\,\mathbb{A}_t^2}]\right)\right) \\ =\; &i\,\mathrm{fp}_{\varepsilon=0}\left(d\,\mathrm{str}\left(\mathbb{A}_t^{2(i-1)}\,\dot{\mathbb{A}}_t\,e^{-\varepsilon\,\mathbb{A}_t^2}\right) - \varepsilon\,d\,\mathrm{str}\left(\mathbb{A}_t^{2i}\,\dot{\mathbb{A}}_t\,e^{-\varepsilon\,\mathbb{A}_t^2}\right)\right) \\ =\; &d\left[i\,\mathrm{fp}_{\varepsilon=0}\left(\mathrm{str}\left(\mathbb{A}_t^{2(i-1)}\,\dot{\mathbb{A}}_t\,e^{-\varepsilon\,\mathbb{A}_t^2}\right) - \varepsilon\,\mathrm{str}\left(\mathbb{A}_t^{2i}\,\dot{\mathbb{A}}_t\,e^{-\varepsilon\,\mathbb{A}_t^2}\right)\right)\right] \end{aligned} \tag{7}$$

is exact. Here we have used the fact that $\frac{d}{dt}\mathbb{A}_t^2 = \dot{\mathbb{A}}_t\,\mathbb{A}_t + \mathbb{A}_t\,\dot{\mathbb{A}}_t = [\mathbb{A}_t, \dot{\mathbb{A}}_t]$, the graded commutator of $\mathbb{A}_t$ with the form Ψdo$-$valued form $\dot{\mathbb{A}}_t$. It follows that the de Rham class of $\mathrm{str}^{\mathbb{A}^2}\left(\mathbb{A}^{2i}\right)$ and hence of $\mathrm{str}^{\mathbb{A}^2}\left(P(\mathbb{A}^2)\right)$ is independent of the choice of connection.

Similarly, since $|\mathbb{A}|^j = \frac{i}{2\pi}\int_\Gamma \lambda^{\frac{j}{2}}\,(\mathbb{A}^2+\pi_A-\lambda)^{-1}\,d\lambda$ and since:

$$\begin{aligned} \frac{d}{dt}(\mathbb{A}_t^2+\pi_A-\lambda)^{-1} &= -(\mathbb{A}_t^2+\pi_{\mathbb{A}}-\lambda)^{-1}\,\frac{d}{dt}\mathbb{A}_t^2\,(\mathbb{A}_t^2+\pi_{\mathbb{A}}-\lambda)^{-1} \\ &= -(\mathbb{A}^2+\pi_{\mathbb{A}}-\lambda)^{-1}\,[\mathbb{A}_t, \dot{\mathbb{A}}_t](\mathbb{A}_t^2+\pi_{\mathbb{A}}-\lambda)^{-1} \\ &= -\left[\mathbb{A}_t, (\mathbb{A}_t^2+\pi_{\mathbb{A}}-\lambda)^{-1}\,\dot{\mathbb{A}}_t\,(\mathbb{A}_t^2+\pi_{\mathbb{A}}-\lambda)^{-1}\right], \end{aligned}$$

it follows that the variation

$$
\begin{aligned}
& \frac{d}{dt}\left(\mathrm{sres}\left(|\mathbb{A}_t|^j\right)\right) \\
= \; & \mathrm{sres}\left(\frac{i}{2\pi}\left[\int_\Gamma \lambda^{\frac{j}{2}} \frac{d}{dt}(\mathbb{A}_t^2+\pi_A-\lambda)^{-1}\, d\lambda\right]\right) \\
= \; & -\frac{i}{2\pi}\,\mathrm{sres}\left(\int_\Gamma \lambda^{\frac{j}{2}}\left[\mathbb{A}_t, (\mathbb{A}_t^2+\pi_{\mathbb{A}}-\lambda)^{-1}\,\dot{\mathbb{A}}_t(\mathbb{A}_t^2+\pi_{\mathbb{A}}-\lambda)^{-1}\right] d\lambda\right) \\
= \; & -\frac{i}{2\pi}\,\mathrm{sres}\left(\int_\Gamma \lambda^{\frac{j}{2}}\left[\mathbb{A}_t+\pi_{\mathbb{A}}, (\mathbb{A}_t^2+\pi_{\mathbb{A}}-\lambda)^{-1}\,\dot{\mathbb{A}}_t(\mathbb{A}_t^2+\pi_{\mathbb{A}}-\lambda)^{-1}\right] d\lambda\right) \\
& \text{since} \quad \mathrm{res}[\pi_{\mathbb{A}}, \cdot] = 0 \\
= \; & -\frac{i}{2\pi}\,\mathrm{sres}\left(\left[\mathbb{A}_t+\pi_{\mathbb{A}}, \int_\Gamma \lambda^{\frac{j}{2}}\;\dot{\mathbb{A}}_t(\mathbb{A}_t^2+\pi_{\mathbb{A}}-\lambda)^{-2}\right]\right) \\
& \text{since} \quad [\mathbb{A}_t+\pi_{\mathbb{A}}, (\mathbb{A}_t^2+\pi_{\mathbb{A}}-\lambda)^{-1}] = 0 \\
= \; & -\frac{i}{2\pi}\,\mathrm{sres}\left(\left[\mathbb{A}_t, \int_\Gamma \lambda^{\frac{j}{2}}\;\dot{\mathbb{A}}_t\,(\mathbb{A}_t^2+\pi_{\mathbb{A}}-\lambda)^{-2}\right]\right) \\
& \text{since} \quad \mathrm{res}[\pi_{\mathbb{A}}, \cdot] = 0 \\
= \; & \frac{j}{2}\frac{i}{2\pi}\,\mathrm{sres}\left(\left[\mathbb{A}_t, \int_\Gamma \lambda^{\frac{j-2}{2}}\;\dot{\mathbb{A}}_t\,(\mathbb{A}_t^2+\pi_{\mathbb{A}}-\lambda)^{-1}\right]\right) \\
= \; & \frac{j}{2}\,\mathrm{sres}\left(\left[\mathbb{A}_t,\;\dot{\mathbb{A}}_t\,(\mathbb{A}_t^2+\pi_{\mathbb{A}})^{\frac{j-2}{2}}\right]\right) \\
= \; & \frac{j}{2}\,d\left(\mathrm{sres}\left(\dot{\mathbb{A}}_t\,|\mathbb{A}_t|^{j-1}\right)\right)
\end{aligned}
\tag{8}
$$

is also exact, which ends the proof of the proposition. □

4.2 Chern-forms associated with superconnections $\mathbb{A} = \nabla + D$

We now specialise to the case $\mathbb{A}_{[2]} = 0$ and consider a superconnection $\mathbb{A} = D + \nabla$ in the graded setup and $\mathbb{A} = \sigma D + \nabla$ in the ungraded setup. The following theorem compares the closed Chern-forms $\mathrm{str}^{\mathbb{A}^2}(\mathbb{A}^{2i})$ with the (non closed in general) weighted Chern forms.

Theorem 1 *In the $\mathbb{Z}_2$-graded set up and provided the superconnection $\mathbb{A} = D + \nabla$, the (closed) Chern-forms $\mathrm{str}^{\mathbb{A}^2}(\mathbb{A}^{2j})_{[2j]}$ differ from the (non closed) Q-weighted Chern forms $\mathrm{str}^Q(\Omega^j)$ by a local map i.e.*

$$
\mathrm{str}^{\mathbb{A}^2}\left(\mathbb{A}^{2j}\right)_{[2j]} - \mathrm{str}^Q(\Omega^j) = \bar{f}_j(\bar{\nabla}).
$$

for some $\bar{f}_j : \mathcal{C}(\bar{P}) \to \Omega(B, \bar{P})$.
In the ungraded setup and provided the superconnection $\mathbb{A} = \sigma D + \nabla$, (closed) Chern-forms $\mathrm{str}^{\mathbb{A}^2}(\mathbb{A}^{2j})_{[2j-1]}$ differ from the (non closed) Q-weighted forms $\mathrm{str}^Q(\Omega^{j-1}[\nabla, \sigma D])$ by a local map i.e.

$$
\mathrm{str}^{\mathbb{A}^2}\left(\mathbb{A}^{2j}\right)_{[2j-1]} - j\,\mathrm{str}^Q(\Omega^{j-1}[\nabla, \sigma D]) = \bar{g}_j(\bar{\nabla})
$$

for some $\bar{g}_j : \mathcal{C}(\bar{P}) \to \Omega(B, \bar{P})$.

Remark 4 *This does not hold anymore if $\mathbb{A}_{[2]} \neq 0$ as can easily be seen from the proof below. When $\mathbb{A}_{[2]} = 0$, on the grounds of this proposition, $\mathrm{str}^{\mathbb{A}^2}\left(\mathbb{A}^{2j}\right)_{[2j]}$ can be interpreted as a renormalised version of $\mathrm{str}^Q(\Omega^j)$.*

Proof: Let us observe in the graded case that since $\mathbb{A}^2 = Q + [\nabla, D] + \Omega$, we have:

$$
\mathrm{str}^Q\left(\mathbb{A}^{2j}\right)_{[2j]} = \mathrm{str}^Q\left(\Omega^j\right),
$$

and similarly in the ungraded case, we have

$$
\mathrm{str}^Q\left(\mathbb{A}^{2j}\right)_{[2j-1]} = j\mathrm{str}^Q\left(\Omega^{j-1}[\nabla, \sigma D]\right).
$$

Using a Campbell-Hausdorff formula for pseudo-differential operators [O] combined with formula (2) extended to form valued weights, we have

$$\begin{aligned}
& \mathrm{str}^{\mathbb{A}^2}\left(\mathbb{A}^{2j}\right)_{[2j]} \\
= \; & \mathrm{str}^{Q}\left(\mathbb{A}^{2j}\right)_{[2j]} + \left(\mathrm{str}^{\mathbb{A}^2}\left(\mathbb{A}^{2j}\right)_{[2j]} - \mathrm{str}^{Q}\left(\mathbb{A}^{2j}\right)_{[2j]}\right) \\
= \; & \mathrm{str}^{Q}\left(\Omega^{j}\right) - \frac{1}{2}\mathrm{sres}\left(\mathbb{A}^{2j}(\log \mathbb{A}^2 - \log Q)\right)_{[2j]} \\
= \; & \mathrm{str}^{Q}(\Omega^{j}) - \frac{1}{2}\mathrm{sres}\left(\mathbb{A}^{2j}\left(\log(1+Q^{-1}([\nabla, D]+\Omega)) + [\log Q, \log(1+Q^{-1}([\nabla, D]+\Omega))] + ...\right)\right)_{[2j]} \\
= \; & \mathrm{str}^{Q}(\Omega^{j-1}) + f_i(\nabla),
\end{aligned}$$

with $f_j(\nabla)$ the Wodzicki residue of a polynomial expression in D, D^{-1} and ∇ of total form degree $2j$. As a Wodzicki residue, it is insensitive to a perturbation of the connection by a smoothing operator so that $f_j(\nabla) = \bar{f}_j(\bar{\nabla})$. This shows that

$$\mathrm{str}^{\mathbb{A}^2}\left(\mathbb{A}^{2i}\right)_{[2j]} - \mathrm{str}^{Q}(\Omega^{j}) = \bar{f}_j(\bar{\nabla})$$

is local. A similar computation shows that

$$\mathrm{str}^{\mathbb{A}^2}\left(\mathbb{A}^{2j}\right)_{[2j-1]} - j\,\mathrm{str}^{Q}(\Omega^{j}\,[\nabla, \sigma\, D]) = \bar{g}_j(\bar{\nabla})$$

is also local. $\square$

4.3 Residue Chern forms as Wodzicki residues

In order to derive an explicit expression for the residue Chern forms in terms of Wodzicki residues, we borrow the following notations from [CoM] and [H]. For A in $Cl(M,E)$ of order a, a given $\Delta \in C\ell(M,E)$ and any $j \in \mathbb{N}$ we set:

$$A^{(j)} := \mathrm{ad}_{\Delta}^{j}(A), \quad \text{where} \quad \mathrm{ad}_{\Delta}(B) = [\Delta, B],$$

so that $A^{(0)} = A, \quad A^{(j+1)} = \mathrm{ad}_{\Delta}(A^{(j)}) = [\Delta, A^{(j)}]$. When Δ of order 2 has scalar leading symbol then $A^{(j)}$ has order $a+j+1$.

Proposition 4 *Let $\mathbb{A}$ be a superconnection associated with an operator D, the square of which has scalar leading symbol. For any positive integer K*

$$\begin{aligned}
\mathrm{sres}\left(|\mathbb{A}|^{2j-1}\right)_{[K]} \; = \; & \sum_{l=0}^{K}\sum_{k_1\geq 0}\cdots\sum_{k_l\geq 0} \frac{\left(\frac{2j-1}{2}\right)\cdots\left(\frac{2j-1}{2}-|k|-l\right)}{(k_1+\cdots+k_l+l)!} c(k_1,\cdots,k_l)\cdot \\
& \cdot\mathrm{sres}\left(\left(\mathbb{A}^2_{[>0]}\right)^{(k_1)}\left(\mathbb{A}^2_{[>0]}\right)^{(k_2)}\cdots\left(\mathbb{A}^2_{[>0]}\right)^{(k_l)}\left(D^2\right)^{\frac{2j-1}{2}-|k|-l}\right)_{[K]},
\end{aligned}$$

where we set $c(k_1) = 1$ for any positive integer k and where, for a multi index $k = (k_1, \cdots, k_l)$ for $j > 1$ we set

$$c(k_1,\cdots,k_l) = \frac{(k_1+\cdots+k_l+l)!}{k_1!\cdots k_j!(k_1+k_2+1)\cdots(k_1+\cdots+k_{l-1}+l)}.$$

In particular,

$$\mathrm{sres}\left(|\mathbb{A}|\right)_{[1]} = \sum_{k\geq 0}\frac{\left(\frac{1}{2}\right)\cdots\left(\frac{1}{2}-k-1\right)}{(k+1)!}\cdot\mathrm{sres}\left(\left(\mathbb{A}^2_{[1]}\right)^{(k)}\left(D^2\right)^{\frac{1}{2}-k-1}\right),$$

Remark 5 *If D is a differential operator then $\mathbb{A}^2$ is a differential operator and $\mathrm{sres}\left(|\mathbb{A}|^{2j}\right) = \mathrm{sres}\left(\left(\mathbb{A}^2\right)^j\right)$ which is why we only consider odd powers.*

Remark 6 *Since the operator order of* $\mathbb{A}^2_{[>0]}$ *is no larger than 1,* $\left(\mathbb{A}^2_{[1]}\right)^{(k)}$ *has order* $\leq 1+k$ *and* $\left(\mathbb{A}^2_{[>0]}\right)^{(k_2)}\cdots\left(\mathbb{A}^2_{[>0]}\right)^{(k_l)}(D^2)^{\frac{2j-1}{2}-|k|-l}$ *has order* $\leq 2j-1-|k|-l$ *which decreases as* $|k|$ *or* l *increases. Thus the Wodzicki residue vanishes for large enough* $|k|$ *or* l *and the seemingly infinite series in the proposition is in fact finite.*

Proof: We introduce notations borrowed from [H] and [CoM]. Let $T \in C\ell(M,E)$ and $T_k, k \in \mathbb{N}$ be operators in $C\ell(M,E)$ with decreasing order in k. Then

$$T \simeq \sum_{k\geq 0} T_k \Longleftrightarrow \forall N \in \mathbb{N}, \exists K(N) \quad T - \sum_{k=0}^{K(N)} T_k \in C\ell^{-N}(M,E).$$

With these notations, for any non negative integer h we have [H] (see the proof of Proposition 4.14)

$$(\lambda - D^2)^{-h} A \simeq \sum_{k\geq 0} \frac{(h+k-1)!}{(h-1)!k!} A^{(k)} (\lambda - D^2)^{-h-k}.$$

As a result, the j-th term in (6) reads:

$$\begin{aligned}
& (\lambda - D^2)^{-1}\, \mathbb{A}^2_{[>0]} \cdots (\lambda - D^2)^{-1}\, \mathbb{A}^2_{[>0]}\, (\lambda - D^2)^{-1} \\
\simeq \;& \sum_{k_1\geq 0} \left(\mathbb{A}^2_{[>0]}\right)^{(k_1)} (\lambda - Q)^{-2-k_1}\, \mathbb{A}^2_{[>0]} \cdots (\lambda - D^2)^{-1}\, \mathbb{A}^2_{[>0]}\, (\lambda - D^2)^{-1} \\
\simeq \;& \sum_{k_1\geq 0} \left(\mathbb{A}^2_{[>0]}\right)^{(k_1)} \sum_{k_2\geq 0} \frac{(-1)^{k_2}(k_1+k_2)!}{k_1!k_2!} \left(\mathbb{A}^2_{[>0]}\right)^{(k_2)} (\lambda - D^2)^{-3-k_1-k_2}\, \mathbb{A}^2_{[>0]} \\
& \cdots (\lambda - D^2)^{-1}\, \mathbb{A}^2_{[>0]}\, (\lambda - D^2)^{-1} \\
\simeq \;& \sum_{k_1\geq 0} \left(\mathbb{A}^2_{[>0]}\right)^{(k_1)} \sum_{k_2\geq 0} \frac{(k_1+k_2)!}{k_1!k_2!} \left(\mathbb{A}^2_{[>0]}\right)^{(k_2)} \sum_{k_3\geq 0} \\
\cdot \;& \frac{(-1)^{k_3}(k_1+k_2+1)!}{(k_1+k_2+1)!k_3!} \left(\mathbb{A}^2_{[>0]}\right)^{(k_3)} (\lambda - D^2)^{-4-k_1-k_2-k_3}\, \mathbb{A}^2_{[>0]} \cdots \\
& \cdots (\lambda - D^2)^{-1}\, \mathbb{A}^2_{[>0]}\, (\lambda - D^2)^{-1} \\
\simeq \;& \sum_{|k|\geq 0} c(k_1, \cdots, k_j) \left(\mathbb{A}^2_{[>0]}\right)^{(k_1)} \left(\mathbb{A}^2_{[>0]}\right)^{(k_2)} \cdots \left(\mathbb{A}^2_{[>0]}\right)^{(k_j)} (\lambda - D^2)^{-|k|-j-1}.
\end{aligned}$$

Letting Γ be a contour around the spectrum $\mathrm{spec}(D^2) \subset \mathbb{R}^+$, it follows that for any positive integer K:

$$\begin{aligned}
& \left((\mathbb{A}^2)^{\frac{2i-1}{2}} \right)_{[K]} \\
= \;& \sum_{j=0}^{K} \frac{1}{2i\pi} \int_\Gamma \lambda^{\frac{2i-1}{2}} \left((\lambda - D^2)^{-1}\, \mathbb{A}^2_{[>0]} \cdots \mathbb{A}^2_{[>0]}\, (\lambda - D^2)^{-1}\, \mathbb{A}^2_{[>0]}\, (\lambda - D^2)^{-1} \right)_{[K]} d\lambda \\
= \;& \sum_{j=0}^{K} \sum_{|k|\geq 0} c(k_1, \cdots, k_j) \left(\left(\mathbb{A}^2_{[>0]}\right)^{(k_1)} \left(\mathbb{A}^2_{[>0]}\right)^{(k_2)} \cdots \left(\mathbb{A}^2_{[>0]}\right)^{(k_j)} \frac{1}{2i\pi} \int_\Gamma \lambda^{\frac{2i-1}{2}} (\lambda - Q)^{-|k|-j-1} d\lambda \right)_{[K]} \\
= \;& \sum_{j=0}^{K} \sum_{|k|\geq 0} \frac{\left(\frac{2i-1}{2}\right) \cdots \left(\frac{2i-1}{2} - |k| - j\right)}{(|k|+j)!} c(k_1, \cdots, k_j) \left(\left(\mathbb{A}^2_{[>0]}\right)^{(k_1)} \left(\mathbb{A}^2_{[>0]}\right)^{(k_2)} \cdots \left(\mathbb{A}^2_{[>0]}\right)^{(k_j)} (D^2)^{2i-1-|k|-j} \right)_{[K]}
\end{aligned}$$

where the last equality follows by integration by parts. Applying the Wodzicki residue yields the result of the proposition. $\square$

5 Getzler's rescaling

Let $\mathcal{E} \to B$ be a Ψdo$-$vector bundle and let $\{D_b, b \in B\}$ be a smooth family of elliptic differential operators parametrised by B acting on the fibres of $\mathcal{E}$.
Getzler's rescaling transforms a homogeneous form $\alpha_{[i]}$ of degree i to the expression

$$\delta_\varepsilon \cdot \alpha_{[i]} \cdot \delta_\varepsilon^{-1} = \frac{\alpha_{[i]}}{\sqrt{\varepsilon}^i},$$

so that a superconnection $\mathbb{A} = \mathbb{A}_{[0]} + \mathbb{A}_{[1]} + \mathbb{A}_{[2]}$ transforms to

$$\tilde{\mathbb{A}}_\varepsilon = \delta_\varepsilon \cdot \mathbb{A} \cdot \delta_\varepsilon^{-1} = \mathbb{A}_{[0]} + \frac{\mathbb{A}_{[1]}}{\sqrt{\varepsilon}} + \frac{\mathbb{A}_{[2]}}{\varepsilon}.$$

Here we allow higher forms in the superconnection, keeping in mind later applications involving the Bismut superconnection for families of Dirac operators. Following the usual conventions, for a given superconnection $\mathbb{A} = \mathbb{A}_{[0]} + \mathbb{A}_{[1]} + \mathbb{A}_{[2]}$ we set:

$$\mathbb{A}_\varepsilon := \sqrt{\varepsilon}\, \tilde{\mathbb{A}}_\varepsilon = \sqrt{\varepsilon}\, \delta_\varepsilon \cdot \mathbb{A} \cdot \delta_\varepsilon^{-1} = \sqrt{\varepsilon}\, \mathbb{A}_{[0]} + \mathbb{A}_{[1]} + \frac{\mathbb{A}_{[2]}}{\sqrt{\varepsilon}}.$$

Remark 7 *Different notations are used in the literature, namely some authors set $t := \sqrt{\varepsilon}$ which leads to (see e.g.[L])*

$$\bar{\mathbb{A}}_t := \mathbb{A}_{t^2} = t\, \mathbb{A}_{[0]} + \mathbb{A}_{[1]} + \frac{\mathbb{A}_{[2]}}{t}, \tag{9}$$

a notation which we shall also use in this paper.

The following result shows how the j-th Chern (resp. residue-) Chern form picks up the $2j$ (resp. $2j-1$-) form degree part of $\mathrm{str}^{\mathbb{A}^2}\left(\mathbb{A}^{2j}\right)$.

Proposition 5 *Let $\mathbb{A}$ be a superconnection associated with a family of elliptic differential operators parametrised by B. Then, with the notations of (9)*

$$\mathrm{str}^{\mathbb{A}^2}\left(\mathbb{A}^{2j}\right)_{[2j]} = \mathrm{fp}_{t=0}\mathrm{str}\left(\bar{\mathbb{A}}_t^{2j}\, e^{-\bar{\mathbb{A}}_t^2}\right)_{[2j]}$$

and

$$\mathrm{sres}\left(|\mathbb{A}|^{2j-1}\right)_{[2j-1]} = \frac{1}{2\sqrt{\pi}}\mathrm{fp}_{t=0}\mathrm{str}\left(\bar{\mathbb{A}}_t^{2j}\, e^{-\bar{\mathbb{A}}_t^2}\right)_{[2j-1]}.$$

Proof: Recall that since D is a differential operator, so is $\mathbb{A}^2$ a differential operator valued form and (see Proposition1)

$$\mathrm{str}^{\mathbb{A}^2}\left(\mathbb{A}^{2j}\right) = \mathrm{fp}_{\varepsilon\to 0}\mathrm{str}\left(\mathbb{A}^{2j} e^{-\varepsilon\,\mathbb{A}^2}\right).$$

On the other hand, for any $t > 0$ we have:

$$\begin{aligned}
\mathrm{str}\left(\bar{\mathbb{A}}_t^{2j}\, e^{-\bar{\mathbb{A}}_t^2}\right)_{[2j]} &= \mathrm{str}\left(\mathbb{A}_{t^2}^{2j}\, e^{-\mathbb{A}_{t^2}^2}\right)_{[2j]} \\
&= \mathrm{str}\left(t^{2j}\, \tilde{\mathbb{A}}_{t^2}^{2j}\, e^{-t^2\,\tilde{\mathbb{A}}_{t^2}^2}\right)_{[2j]} \\
&= \mathrm{str}\left(t^{2j}\, \delta_{t^2}\, \mathbb{A}^{2j}\, \delta_{t^2}^{-1}\, e^{-t^2\,\delta_{t^2}\,\mathbb{A}^2\delta_{t^2}^{-1}}\right)_{[2j]} \\
&= \mathrm{str}\left(t^{2j}\, \delta_{t^2}\, \left(\mathbb{A}^{2j}\, e^{-t^2\,\mathbb{A}^2}\right)\delta_{t^2}^{-1}\right)_{[2j]} \\
&= \mathrm{str}\left(\mathbb{A}^{2j}\, e^{-t^2\,\mathbb{A}^2}\right)_{[2j]}.
\end{aligned}$$

Since the r.h. side is of the type $\mathrm{str}\left(\mathbb{A}^{2j}\, e^{-\varepsilon\,\mathbb{A}^2}\right)$ with $\mathbb{A}^2$ an elliptic differential operator valued form and $\mathbb{A}^{2j} \in \Omega(B, C\ell(\mathcal{E}))$, it has a known asymptotic expansion at 0 and taking finite parts when $t \to 0$

yields the first part of the proposition.

Similarly, since D is a differential operator we have

$$\mathrm{sres}\left(|\mathbb{A}|^{2j-1}\right)=\frac{1}{2\sqrt{\pi}}\mathrm{fp}_{t=0}\left(t\,\mathrm{str}\left(\mathbb{A}^{2i}\,e^{-t^2\,\mathbb{A}^2}\right)\right)$$

and for any $t>0$

$$\begin{aligned}\mathrm{str}\left(\bar{\mathbb{A}}_t^{2j}\,e^{-\bar{\mathbb{A}}_t^2}\right)_{[2j-1]} &= \mathrm{str}\left(\mathbb{A}_{t^2}^{2j}\,e^{-\mathbb{A}_{t^2}^2}\right)_{[2j-1]}\\ &= \mathrm{str}\left(t^{2j}\,\tilde{\mathbb{A}}_{t^2}^{2j}\,e^{-t^2\,\tilde{\mathbb{A}}_{t^2}^2}\right)_{[2j-1]}\\ &= \mathrm{str}\left(t^{2j}\,\delta_{t^2}\,\mathbb{A}^{2j}\,\delta_{t^2}^{-1}\,e^{-t^2\,\delta_{t^2}\,\mathbb{A}^2\delta_{t^2}^{-1}}\right)_{[2j-1]}\\ &= \mathrm{str}\left(t^{2j}\,\delta_{t^2}\,\left(\mathbb{A}^{2j}\,e^{-t^2\,\mathbb{A}^2}\right)\delta_{t^2}^{-1}\right)_{[2j-1]}\\ &= t\,\mathrm{str}\left(\mathbb{A}^{2j}\,e^{-t^2\,\mathbb{A}^2}\right)_{[2j-1]}.\end{aligned}$$

As before, since the r.h.s. is of the type $\mathrm{str}\left(\mathbb{A}^{2j}\,e^{-\varepsilon\,\mathbb{A}^2}\right)$ with $\mathbb{A}^2$ an elliptic differential operator valued form and $\mathbb{A}^{2j}\in\Omega(B,C\ell(\mathcal{E}))$, it has a known asymptotic expansion at 0 and taking finite parts when $t\to 0$ yields the first part of the lemma. Taking finite parts when $t\to 0$ therefore yields the second part of the lemma. □

6 Superconnections associated with Dirac operators

We now specialise to Chern forms built from superconnections associated with families of Dirac operators. Let $\pi:\mathbb{M}\to B$ be a smooth fibration of closed spin manifolds with fibre M and $\mathbb{E}\to B$ a Clifford bundle with an associated family of Dirac operators parametrised by B. The vector bundle $\mathcal{E}:=\pi_*\mathbb{E}$ is an (infinite rank) Ψdo$-$vector bundle with fibres modelled on $C^\infty(\mathbb{M}/B,\mathbb{E}_{\mathbb{M}/B})$. According to whether the manifolds are even or odd dimensional, $\mathcal{E}$ will be $\mathbb{Z}_2$-graded or not.
The vertical Riemannian metric g^{TM} and hermitian metric $h^{\mathbb{E}}$ on $\mathbb{E}$ induce an L^2-inner product on $\mathcal{E}$. From a connection $\nabla^{\mathbb{E}}$ on $\mathbb{E}$ compatible with $h^{\mathbb{E}}$, one can build a connection $\tilde{\nabla}^{\mathcal{E}}_X\sigma:=\nabla^{\mathbb{E}}_{\tilde{X}}\sigma(b)$ on $\mathcal{E}=\pi_*\mathbb{E}$ where $\tilde{X}$ is the horizontal lift of $X\in T_bB$ and from there a unitary connection $\nabla^{\mathcal{E}}$ on $\mathcal{E}$.
The corresponding Bismut superconnection associated with this fibration reads:

$$\mathbb{A}=D+\nabla^{\mathcal{E}}+c(T)\quad\text{even}\quad\text{case},\qquad \mathbb{A}=\sigma\,D+\nabla^{\mathcal{E}}+\sigma\,c(T)\quad\text{odd}\quad\text{case},$$

where $T\in\Omega^2(\mathbb{M},TM)$ is the curvature of the horizontal distribution on $\mathbb{M}$ and c the Clifford multiplication.
Along the lines of the heat-kernel proof of the index theorem we introduce the kernel $k_\varepsilon(\mathbb{A}^2)$ of $e^{-\varepsilon\,\mathbb{A}^2}$ for some $\varepsilon>0$. Since D is a family of Dirac operators, we have (see e.g. chap. 10 in [BGV] in the even dimensional case and [BC] in the odd dimensional case)

$$k_\varepsilon(\mathbb{A}^2)(x,x)\sim_{\varepsilon\to 0}\frac{1}{(4\pi\varepsilon)^{\frac{n}{2}}}\sum_{j=0}^{\infty}\varepsilon^j k_j(\mathbb{A}^2)(x,x)\tag{10}$$

Proposition 6 *Let $\mathbb{A}$ be a superconnection adapted to a smooth family of Dirac operators parametrised by B.*

1. *The j-th Chern form associated with $\mathbb{A}$ is given by an integration along fiber of $\mathbb{M}$:*

$$\mathrm{str}^{\mathbb{A}^2}\left(\mathbb{A}^{2j}\right)=\frac{(-1)^j j!}{(4\pi)^{\frac{n}{2}}}\int_{\mathbb{M}/B}\mathrm{str}(k_{j+\frac{n}{2}}(\mathbb{A}^2)).$$

2. *If the kernel of D has constant dimension, the j-th residue Chern form associated with $\mathbb{A}$ reads:*

$$\sqrt{\pi}\,\mathrm{sres}\left(|\mathbb{A}|^{2j-1}\right) = \frac{(-1)^j(2j-1)!!}{(4\pi)^{\frac{n}{2}}2^{j-1}}\int_{M/B}\mathrm{str}(k_{j+\frac{n-1}{2}}(\mathbb{A}^2)).$$

Here $|\mathbb{A}|^{2j-1}$ is defined as before by the contour integral

$$|\mathbb{A}|^{2j-1} = \frac{i}{2\pi}\int_\Gamma \lambda^{\frac{2j-1}{2}}\left(\mathbb{A}^2 + \pi_{\mathbb{A}} - \lambda\right)^{-1} d\lambda.$$

Remark 8 *It follows from this last formula that adding a smoothing* Ψdo*−valued zero form to* $\mathbb{A}^2$ *does not affect the residue form* $\mathrm{sres}\left(|\mathbb{A}|^{2j-1}\right)_{[2j-1]}$ *so that one expects formulae for the residue Chern form to be independent of* $\pi_{\mathbb{A}}$*.*

Proof: The trace under the integral sign is just the matrix trace for endomorphims of a finite rank vector bundle E whereas on the left-hand-side the trace is computed in the Hilbert space of square-integrable sections of E over M.

By the above remark, $\mathbb{A}$ being a differential operator, it has vanishing Wodzicki residue and we have:

$$\mathrm{str}^{\mathbb{A}^2}\left(\mathbb{A}^{2j}\right) = \mathrm{fp}_{\varepsilon=0}\mathrm{str}\left(\mathbb{A}^{2j}\,e^{-\varepsilon\,\mathbb{A}^2}\right) = (-1)^j\mathrm{fp}_{\varepsilon=0}\left(\partial_\varepsilon^j\mathrm{str}\left(e^{-\varepsilon\,\mathbb{A}^2}\right)_{|\varepsilon=0}\right).$$

By equation (10), this yields

$$\begin{aligned}
\mathrm{str}^{\mathbb{A}^2}\left(\mathbb{A}^{2j}\right) &= (-1)^j\mathrm{fp}_{\varepsilon=0}\left(\partial_\varepsilon^j\mathrm{str}\left(e^{-\varepsilon\,\mathbb{A}^2}\right)\right)\\
&= (-1)^j\,\mathrm{fp}_{\varepsilon=0}\left(\partial_\varepsilon^j\int_{M/B}\mathrm{str}\left(k_\varepsilon(\mathbb{A}^2)\right)\right)\\
&= \frac{(-1)^j}{(4\pi)^{\frac{n}{2}}}\,\mathrm{fp}_{\varepsilon=0}\left(\partial_\varepsilon^i\sum_{j=0}^\infty\varepsilon^{j-\frac{n}{2}}\int_{M/B}\mathrm{str}(k_j(\mathbb{A}^2))\right)\\
&= \frac{(-1)^i}{(4\pi)^{\frac{n}{2}}}\,\mathrm{fp}_{\varepsilon=0}\left(\sum_{l=0}^\infty(l-\frac{n}{2})\cdots(l-\frac{n}{2}-j+1)\varepsilon^{l-\frac{n}{2}-j}\int_{M/B}\mathrm{str}(k_l(\mathbb{A}^2)\right)\\
&= \frac{(-1)^j j!}{(4\pi)^{\frac{n}{2}}}\int_{M/B}\mathrm{str}(k_{j+\frac{n}{2}}(\mathbb{A}^2)).
\end{aligned}$$

This proves the first part of the proposition. On the other hand, again by (10) we have:

$$\begin{aligned}
\mathrm{fp}_{\varepsilon=0}\mathrm{str}\left(\sqrt{\varepsilon}\left(\mathbb{A}^{2j}e^{-\varepsilon\,\mathbb{A}^2}\right)\right) &= (-1)^j\mathrm{fp}_{\varepsilon=0}\left(\sqrt{\varepsilon}\,\partial_\varepsilon^j\mathrm{str}\left(e^{-\varepsilon\,\mathbb{A}^2}\right)\right)\\
&= (-1)^j,\mathrm{fp}_{\varepsilon=0}\left(\sqrt{\varepsilon}\,\partial_\varepsilon^j\int_{M/B}\mathrm{str}\left(k_\varepsilon(\mathbb{A}^2)\right)\right)\\
&= \frac{(-1)^j}{(4\pi)^{\frac{n}{2}}}\,\mathrm{fp}_{\varepsilon=0}\left(\sqrt{\varepsilon}\,\partial_\varepsilon^j\sum_{l=0}^\infty\varepsilon^{l-\frac{n}{2}}\int_{M/B}\mathrm{str}(k_l(\mathbb{A}^2))\right)\\
&= \frac{(-1)^j}{(4\pi)^{\frac{n}{2}}}\,\mathrm{fp}_{\varepsilon=0}\left(\sum_{l=0}^\infty(l-\frac{n}{2})\cdots(l-\frac{n}{2}-j+1)\varepsilon^{l-\frac{n-1}{2}-j}\int_{M/B}\mathrm{str}(k_l(\mathbb{A}^2))\right)\\
&= \frac{(-1)^j(j-\frac{1}{2})(j-\frac{3}{2})\cdots\frac{1}{2}}{(4\pi)^{\frac{n}{2}}}\int_{M/B}\mathrm{str}(k_{j+\frac{n-1}{2}}(\mathbb{A}^2))\\
&= \frac{(-1)^j(2j-1)!!}{(4\pi)^{\frac{n}{2}}2^{j-1}}\int_{M/B}\mathrm{str}(k_{j+\frac{n-1}{2}}(\mathbb{A}^2)).
\end{aligned}$$

On the other hand, Lemma 1 applied to $g(\varepsilon) = \mathrm{str}\left(\mathbb{A}^{2j} e^{-\varepsilon\,\mathbb{A}^2}\right)$ then yields:

$$\begin{aligned}
\mathrm{fp}_{\varepsilon=0}\left(\sqrt{\varepsilon}\,\mathrm{str}\left(\mathbb{A}^{2j} e^{-\varepsilon\,\mathbb{A}^2}\right)\right) &= \Gamma(\frac{1}{2})\,\mathrm{res}_{z=0}\,\mathrm{str}\left(\mathbb{A}^{2j}(\mathbb{A}^2)^{-z-\frac{1}{2}}\right)\\
&= 2\sqrt{\pi}\,\mathrm{sres}\left(\mathbb{A}^{2j}(\mathbb{A}^2)^{-\frac{1}{2}}\right)
\end{aligned}$$

so that,

$$\sqrt{\pi}\,\mathrm{sres}\left(\mathbb{A}^{2j}\left(\mathbb{A}^2\right)^{-\frac{1}{2}}\right) = \frac{(-1)^j(2j-1)!!}{(4\pi)^{\frac{n}{2}}2^j}\int_{M/B}\mathrm{str}(k_{j+\frac{n-1}{2}}(\mathbb{A}^2))$$

which proves the second part of the the proposition. □

The following result relates the j-th (resp. residue) Chern form with the $2j$-th (resp. $2j-1$-th) form degree part of the Chern character $\lim_{t\to 0}\mathrm{ch}(\mathbb{A}_t)$.

Theorem 2 *In the $\mathbb{Z}_2$-graded case the j-th Chern form associated with a superconnection $\mathbb{A}$ reads:*

$$\begin{aligned}
\mathrm{str}^{\mathbb{A}^2}\left(\mathbb{A}^{2j}\right)_{[2j]} &= \frac{(-1)^j j!}{(2i\pi)^{\frac{n}{2}}}\left(\int_{M/B}\hat{A}(M/B)\wedge\mathrm{ch}(E_{M/B})\right)_{[2j]}\\
&= (-1)^j j!\left(\lim_{t\to 0}\mathrm{ch}(\mathbb{A}_t)\right)_{[2j]}. \qquad (11)
\end{aligned}$$

In the ungraded case the j-th residue Chern form associated with the superconnection with kernel of D of constant dimension reads:

$$\begin{aligned}
\mathrm{sres}\left(|\mathbb{A}|^{2j-1}\right)_{[2j-1]} &= \frac{(-1)^j(2j-1)!!}{(2i\pi)^{\frac{n+1}{2}}2^{j-1}}\left(\int_{M/B}\hat{A}(M/B)\wedge\mathrm{ch}(E_{M/B}))\right)_{[2j-1]}\\
&= \frac{(-1)^j(2j-1)!!}{2^{j-1}\sqrt{\pi}}\left(\lim_{t\to 0}\mathrm{ch}(\mathbb{A}_t)\right)_{[2j-1]}. \qquad (12)
\end{aligned}$$

Proof: As in [BGV] par. 10.4, using the asymptotic expansion of the kernel $k_t(x,x)$ of the heat-operator $e^{-t\,\mathbb{A}^2}$:

$$k_t(x,x) \sim_{t\to 0}\frac{1}{(4\pi t)^{\frac{n}{2}}}\sum_{j=0}^{\infty}t^j\,k_j(x,x)$$

we have:

$$\begin{aligned}
\mathrm{ch}(\mathbb{A}_t) &= \delta_t\left(\mathrm{str}(e^{-t\,\mathbb{A}^2})\right)\\
&\sim_{t\to 0} (4\pi t)^{-\frac{n}{2}}\sum_j t^j\int_{M/B}\delta_t\left(\mathrm{str}(k_j(\mathbb{A}^2))\right)\\
&\sim_{t\to 0} (4\pi)^{-\frac{n}{2}}\sum_{j,p}t^{j-(n+p)/2}\left(\int_{M/B}\mathrm{str}\left(k_j(\mathbb{A}^2)\right)\right)_{[p]},
\end{aligned}$$

so that

$$\mathrm{fp}_{t=0}\mathrm{ch}(\mathbb{A}_t)_{[p]} = (4\pi)^{-\frac{n}{2}}\left(\int_{M/B}\mathrm{str}\left(k_{\frac{p+n}{2}}(\mathbb{A}^2)\right)\right)_{[p]}. \qquad (13)$$

- **$\mathbb{Z}_2$-graded case.** The family index theorem [B] (see also Theorem 10.23 in [BGV]) yields the existence of the limit as $t\to 0$ and

$$\lim_{t\to 0}\mathrm{ch}(\mathbb{A}_t) = (2i\pi)^{-\frac{n}{2}}\int_{M/B}\hat{A}(M/B)\wedge\mathrm{ch}(E_{M/B}).$$

Combining these two facts leads to:

$$\left(\int_{\mathrm{M}/B} \mathrm{str}\left(k_{\frac{n+2j}{2}}(\mathbb{A}^2)\right)\right)_{[2j]} = \frac{(4\pi)^{\frac{n}{2}}}{(2i\pi)^{\frac{n}{2}}}\left(\int_{\mathrm{M}/B} \hat{A}(\mathrm{M}/B)\wedge \mathrm{ch}(\mathrm{E}_{\mathrm{M}/B})\right)_{[2j]}.$$

Inserting this in Proposition 6 yields

$$\mathrm{str}^{\mathbb{A}^2}\left(\mathbb{A}^{2j}\right)_{[2j]} = \frac{(-1)^j j!}{(2i\pi)^{\frac{n}{2}}}\left(\int_{\mathrm{M}/B} \hat{A}(\mathrm{M}/B)\wedge \mathrm{ch}(\mathrm{E}_{\mathrm{M}/B})\right)_{[2j]}$$

and hence (11).

- **Ungraded case.** The family index theorem yields the existence of the limit and [BC].

$$\lim_{t\to 0} \mathrm{ch}(\mathbb{A}_t) = \frac{\sqrt{\pi}}{(2\pi i)^{\frac{n+1}{2}}}\int_{\mathrm{M}/B} \hat{A}(\mathrm{M}/B)\wedge \mathrm{ch}(\mathrm{E}_{\mathrm{M}/B}).$$

Combined with (13) this yields:

$$\left(\int_{\mathrm{M}/B} \mathrm{str}\left(k_{\frac{n+2j-1}{2}}(\mathbb{A}^2)\right)\right)_{[2j-1]} = \sqrt{\pi}\,\frac{(4\pi)^{\frac{n}{2}}}{(2i\pi)^{\frac{n+1}{2}}}\left(\int_{\mathrm{M}/B} \hat{A}(\mathrm{M}/B)\wedge \mathrm{ch}(\mathrm{E}_{\mathrm{M}/B})\right)_{[2j-1]}.$$

Inserting this in Proposition 6 gives:

$$\mathrm{sres}\left(|\mathbb{A}|^{2j-1}\right)_{[2j-1]} = \frac{(-1)^j(2j-1)!!}{(2i\pi)^{\frac{n+1}{2}}\,2^{j-1}}\left(\int_{\mathrm{M}/B} \hat{A}(\mathrm{M}/B)\wedge \mathrm{ch}(\mathrm{E}_{\mathrm{M}/B}))\right)_{[2j-1]}$$

and hence (12).

□

Corollary 1 *Whenever the fibration* $\mathrm{M}\to B$ *is trivial, then*

$$\mathrm{tr}^Q(\Omega^j) - \frac{(-1)^j j!}{(2i\pi)^{\frac{n}{2}}}\left(\int_{\mathrm{M}/B} \hat{A}(\mathrm{M}/B)\wedge \mathrm{ch}(\mathrm{E})\right)_{[2j]} = \bar{f}_j(\bar{\nabla}). \tag{14}$$

is local in the sense of the above definition.

Example 3 *In the* $\mathbb{Z}_2$*-graded case,* $\frac{1}{(2i\pi)^{\frac{n}{2}}}\left(\int_{\mathrm{M}/B} \hat{A}(\mathrm{M}/B)\wedge \mathrm{ch}(\mathrm{E}_{\mathrm{M}/B})\right)_{[2]}$ *corresponds to the curvature on the determinant line bundle associated with a family of Dirac operators [BF]. The formula corresponding to* $j=1$ *in the above theorem*

$$\mathrm{str}^{\mathbb{A}^2}\left(\mathbb{A}^2\right)_{[2]} = -\frac{1}{(2i\,\pi)^{\frac{n}{2}}}\left(\int_{\mathrm{M}/B} \hat{A}(\mathrm{M}/B)\wedge \mathrm{ch}(\mathrm{E}_{\mathrm{M}/B})\right)_{[2]}$$

expresses the curvature on the determinant bundle as $-\frac{1}{(2i)^{\frac{n}{2}}}$ *times the degree 2 part of the first Chern form associated with the superconnection* $\mathbb{A}$*, thereby generalising the relation that holds in finite dimensions (corresponding to the case* $n=0$ *of a zero dimensional fibre* M*) relating the first Chern form on a finite rank supervector bundle with minus the curvature on its determinant bundle (see [PR] for a discussion concerning this relation).*

Example 4 *In the ungraded case,* $\left(\int_{\mathbb{M}/B} \hat{A} \wedge \mathrm{ch}(\mathbb{E}_{\mathbb{M}/B})\right)_{[2j-1]}$ *corresponds to the curvature of a gerbe with connection associated with the family of Dirac operators [CM],[EM],[L]. The formula obtained in the above theorem for* $j = 2$

$$\mathrm{sres}\left(\mathbb{A}^2\,|\,\mathbb{A}|\right)_{[3]} = \frac{3}{2\,(2i\pi)^{\frac{n+1}{2}}}\left(\int_{\mathbb{M}/B} \hat{A} \wedge \mathrm{ch}(\mathbb{E}_{\mathbb{M}/B})\right)_{[3]} \tag{15}$$

where we have set $|\mathbb{A}| = (\mathbb{A}^2)^{\frac{1}{2}}$ *relates this curvature with the degree* 3 *part of the residue Chern form* $\mathrm{sres}\left(\mathbb{A}^2\,|\,\mathbb{A}|\right)_{[3]}$.

7 Transgressed residue Chern forms

Let as before $\pi : \mathbb{M} \to B$ be a smooth fibration of closed odd dimensional spin manifolds with fibre M and $\mathbb{E} \to B$ a Clifford bundle. The vector bundle $\mathcal{E} := \pi_* \mathbb{E}$ is an (infinite rank) Ψdo$-$vector bundle with fibres modelled on $C^\infty(\mathbb{M}/B, \mathbb{E}_{\mathbb{M}/B})$. Let $\{D_b, b \in B\}$ be a smooth family of Dirac operators associated with this fibration.
We need to work with invertible operators and introduce for this purpose a covering of B by open sets $\{U_\lambda\}_{\lambda\in\mathbb{R}}$ with the property that for any $b \in U_\lambda$ the (discrete) spectrum of D_b does not contain λ. Then $D(\lambda) = D - \lambda I$ is a family of Dirac type operators which is everywhere invertible on U_λ and

$$\mathbb{A}_\lambda := \sigma\, D(\lambda) + \nabla^{\mathcal{E}} + \sigma\,\frac{c(T)}{4}$$

is a superconnection associated with $D(\lambda)$.

With the notations of (9) we set

$$\bar{\mathbb{A}}_t := t\,\sigma\, D + \nabla^{\mathcal{E}} + \sigma\frac{c(T)}{4t},$$

where σ is the grading, which defines a smooth family of superconnections adapted to D.
Let for $t > 0$

$$\bar{\mathbb{A}}_{\lambda,t} := \sigma\, t\, D(\lambda) + \nabla^{\mathcal{E}} + \sigma\,\frac{c(T)}{4t},$$

which defines a smooth family of superconnections adapted to $D(\lambda)$. The following technical result will be useful for what follows.

Lemma 2 *Let* γ *be a differential operator valued form on* B.

1. *The function*

$$t \mapsto \mathrm{str}\left(\gamma\, e^{-\bar{\mathbb{A}}^2_{\lambda,t}}\right)$$

decreases faster than any power of t *as* $t \to \infty$. *As* $t \to 0$ *it behaves as a finite linear combination of expressions* $\sum_{j=0}^\infty \alpha_j t^{j-\delta}$ *for some integer* δ *and complex numbers* α_j, β_k.

2. *If* γ *is an* **even** *form, then*

   (a) $\mathrm{str}\left(\sigma\,\gamma\, e^{-\bar{\mathbb{A}}^2_{\lambda,t}}\right)$ *only involves odd powers of* t *and* t^{-1}. *In particular,*

$$\mathrm{fp}_{t=0}\mathrm{str}\left(\sigma\,\gamma\, e^{-\bar{\mathbb{A}}^2_{\lambda,t}}\right) = 0.$$

   (b) For any non negative integer j

$$\left(\mathrm{str}\left(\sigma\,\gamma\, e^{-\bar{\mathbb{A}}^2_{\lambda,t}}\right)\right)_{[2j]} = 0 \quad \forall \quad t > 0.$$

Remark 9 *The lemma easily extends to* Ψdo$-$*valued forms if we allow for logarithmic divergences in* t *in which case* δ *is a real number.*

Proof:

1. Let us first introduce notations similar to notations of [H]. Let $T \in C\ell(M,E)$ and $T_k, k \in \mathbb{N}$ be operators in $C\ell(M,E)$ with decreasing order in k. Then

$$\begin{aligned} T &\sim \sum_{k\geq 0} T_k \\ \Longleftrightarrow \quad & \exists C \in C\ell(M,E) \quad \text{invertible, s.t.} \quad \forall N \in \mathbb{N}, \exists K(N) \\ & \left(T - \sum_{k=0}^{K(N)} T_k\right) C \in C\ell^{-N}(M,E). \end{aligned} \tag{16}$$

In the sequel, the operator $e^{-t^2 D^2}$ plays the role of the invertible operator C.
We also need to extend to Ψdo$-$valued forms, notations previously used for ordinary classical pseudo-differential operators. For $\beta \in \Omega\left((B, C\ell\left(\mathcal{E}\right)\right)$ and any $j \in \mathbb{N}$ we set:

$$\alpha^{(j)}(\beta) := \mathrm{ad}_{D^2}^j(\beta), \quad \text{where} \quad \mathrm{ad}_{D^2}(\beta) = [D^2, \beta],$$

so that $\beta^{(0)} = \beta, \quad \beta^{(j+1)} = \mathrm{ad}_{D^2}(\beta^{(j)}) = [D^2, \beta^{(j)}]$.

Since $\bar{\mathbb{A}}_{\lambda,t} = t\,\sigma\, D(\lambda) + \left(\bar{\mathbb{A}}_{\lambda,t}\right)_{[>0]}$ we have

$$\begin{aligned} \bar{\mathbb{A}}_{\lambda,t}^2 &= t^2\, D(\lambda)^2 + \left(\bar{\mathbb{A}}_{\lambda,t}^2\right)_{[>0]} \\ &= t^2\, D(\lambda)^2 + \sigma\left[t\,[\nabla^{\mathcal{E}}, D(\lambda)] + \frac{[\nabla^{\mathcal{E}}, c(T)]}{4t}\right] + \frac{1}{4}[D(\lambda), c(T)] + \frac{c^2(T)}{16\, t^2} + \Omega^{\mathcal{E}} \\ &= t^2\, D(\lambda)^2 + \sigma\, \left(\bar{\mathbb{A}}_{\lambda,t}\right)_{[>0,od]} + \left(\bar{\mathbb{A}}_{\lambda,t}\right)_{[>0,ev]} \end{aligned}$$

where we have set $\left(\bar{\mathbb{A}}_{\lambda,t}\right)_{[>0,od]} := t\,[\nabla^{\mathcal{E}}, D(\lambda)] + \frac{[\nabla^{\mathcal{E}}, c(T)]}{4t}$ which only involves odd powers of t and t^{-1}.
Duhamel's formula then yields (see e.g. [H]):

$$\begin{aligned} e^{-\bar{\mathbb{A}}_{\lambda,t}^2} &= (-1)^n \int_{\Delta_l} e^{-u_0\, t^2 D^2} \left(\bar{\mathbb{A}}_{\lambda,t}^2\right)_{[>0]} \cdots e^{-u_{l-1}\, t^2 D(\lambda)^2} \left(\bar{\mathbb{A}}_{\lambda,t}^2\right)_{[>0]} e^{-u_l\, t^2 D(\lambda)^2}\, du_1 \cdots du_l \\ &\sim \sum_{|k|\geq 0} \frac{(-1)^{|k|} t^{2|k|}\, c(k)}{(|k|+n)!} \left(\left(\bar{\mathbb{A}}_{\lambda,t}^2\right)_{[>0]}\right)^{(k_1)} \cdots \left(\left(\bar{\mathbb{A}}_{\lambda,t}^2\right)_{[>0]}\right)^{(k_l)} e^{-t^2 D(\lambda)^2}. \end{aligned}$$

Here $\Delta_l := \{(u_0, \cdots, u_l), u_i \geq 0, \quad \sum_{i=0}^l u_i = 1\}$ is the unit simplex and with the coefficient $c(k)$ as previously defined.
Since $\left(\bar{\mathbb{A}}_{\lambda,t}^2\right)_{[>0]}$ has positive degree, only a finite number of terms of the sum will contribute for a fixed form degree.
Now, for a differential operator C of order c, the map $t \mapsto \mathrm{str}\left(C\, e^{-t^2 D^2}\right)$ decreases faster than any power of t at infinity and behaves asymptotically as follows as $t \to 0$:

$$\mathrm{str}\left(C\, e^{-t^2 D^2}\right) \sim_{t\to 0} \sum_{j=0}^{\infty} \alpha_j t^{j-c-n} \tag{17}$$

where c is the order of C and n the dimension of the manifold.
Since expressions of the type $\left(\bar{\mathbb{A}}_{\lambda,t}^2\right)_{[>0]}$ are linear combinations of differential operator valued forms with coefficients given by powers of t, for any Ψdo$-$valued form γ on B

$$\mathrm{str}\left(\gamma\, \left(\left(\bar{\mathbb{A}}_t^2\right)_{[>0]}\right)^{(k_1)} \cdots \left(\left(\bar{\mathbb{A}}_t^2\right)_{[>0]}\right)^{(k_n)} e^{-t^2 D^2}\right)$$

has the expected asymptotic behaviour at 0 and at ∞. It follows that so does str $\left(\gamma\, e^{-\,\bar{\mathbb{A}}_t^2}\right)$ have a similar asymptotic behaviour. This ends the proof of the first part of the lemma.

2. The second part of the lemma requires a closer look at the expressions involved. Since

$$\left(\bar{\mathbb{A}}_{\lambda,t}\right)_{[>0]} = \sigma\, \left(\bar{\mathbb{A}}_{\lambda,t}\right)_{[>0,od]} + \left(\bar{\mathbb{A}}_{\lambda,t}\right)_{[>0,ev]}$$

and since str vanishes on terms of the type $\sigma\beta$, in the expression

$$\begin{aligned} &\mathrm{str}\left(\sigma\,\gamma\, e^{-\,\bar{\mathbb{A}}^2_{\lambda,t}}\right) \\ = &\sum_{|k|\geq 0} \frac{(-1)^{|k|} t^{2|k|}\, c(k)}{(|k|+n)!} \mathrm{str}\left(\sigma\,\gamma\, \left(\left(\bar{\mathbb{A}}^2_{\lambda,t}\right)_{[>0]}\right)^{(k_1)} \cdots \left(\left(\bar{\mathbb{A}}^2_{\lambda,t}\right)_{[>0]}\right)^{(k_l)} e^{-t^2 D(\lambda)^2}\right) \end{aligned}$$

(which we recall only contains a finite number of terms for fixed form degree) only those terms will remain that involve an odd number of expressions of the type $\left(\bar{\mathbb{A}}_{\lambda,t}\right)_{[>0,od]}$ and hence odd powers of t and t^{-1}. Since γ is assumed to be of even degree, it follows that the total expression str $\left(\sigma\,\gamma\, e^{-\,\bar{\mathbb{A}}^2_{\lambda,t}}\right)$ is an odd degree form which only involves odd powers of t and t^{-1} so that

$$\mathrm{fp}_{t=0}\left(\mathrm{str}\left(\sigma\,\gamma\, e^{-\,\bar{\mathbb{A}}^2_{\lambda,t}}\right)\right) = 0; \quad \left(\mathrm{str}\left(\sigma\,\gamma\, e^{-\,\bar{\mathbb{A}}^2_{\lambda,t}}\right)\right)_{[2j]} = 0 \quad \forall t>0.$$

This proves the second part of the lemma.

□

The following theorem provides a transgression formula for the residue Chern forms sres $\left(|\,\mathbb{A}|^{2j-1}\right)_{[2j-1]}$.

Theorem 3 *On every open subset U_λ, the following transgression formula holds:*

$$\mathrm{sres}\left(|\,\mathbb{A}_\lambda|^{2j-1}\right)_{[2j-1]} = a_j \cdot d\,\left(\tilde{\eta}_\lambda\right)_{[2j-2]},$$

where d is the exterior differential, $a_j := \frac{(-1)^j (2j-1)!!}{2^{j-1}}$ and

$$\tilde{\eta}_\lambda = \mathrm{fp}_{t=0} \int_t^\infty \mathrm{str}\left[\frac{d}{ds}\,\bar{\mathbb{A}}_{\lambda,s}\, e^{-\,\bar{\mathbb{A}}^2_{\lambda,s}}\right] ds$$

is the η invariant associated with $D(\lambda)$ (see [BC], [L]).

Proof: We first show that sres $\left(|\,\mathbb{A}|^{2j-1}\right)_{[2j-1]} =$ sres $\left(|\,\mathbb{A}_\lambda|^{2j-1}\right)_{[2j-1]}$ and then show a transgression formula for sres $\left(|\,\mathbb{A}_\lambda|^{2j-1}\right)_{[2j-1]}$.

1. Let us consider a smooth family $D(\lambda)(\varepsilon) := D - \varepsilon\,\lambda\, I$ of first order elliptic differential operators interpolating D and $D(\lambda)$ between 0 and 1 and the corresponding superconnections

$$\bar{\mathbb{A}}_{\lambda,t}(\varepsilon) := \mathbb{A}_{\lambda,t} - \varepsilon\,\sigma\,\lambda\, I.$$

Differentiating w.r. to ε we have:

$$\begin{aligned} \frac{d}{d\varepsilon}\mathrm{str}\left(e^{-\,\bar{\mathbb{A}}^2_{\lambda,t}(\varepsilon)}\right) &= -\mathrm{str}\left([\,\bar{\mathbb{A}}_{\lambda,t}(\varepsilon), \frac{d}{d\varepsilon}\,\bar{\mathbb{A}}_{\lambda,t}(\varepsilon)]\; e^{-\,\bar{\mathbb{A}}^2_{\lambda,t}(\varepsilon)}\right) \\ &= -\,\mathrm{str}\left([\,\bar{\mathbb{A}}_{\lambda,t}, \frac{d}{d\varepsilon}\,\bar{\mathbb{A}}_{\lambda,t}(\varepsilon)\; e^{-\,\bar{\mathbb{A}}^2_{\lambda,t}(\varepsilon)}]\right) \\ &= -d\,\mathrm{str}\left(\frac{d}{d\varepsilon}\,\bar{\mathbb{A}}_{\lambda,t}(\varepsilon)\, e^{-\,\bar{\mathbb{A}}^2_{\lambda,t}(\varepsilon)}\right) \\ &= \lambda\, d\,\mathrm{str}\left(\sigma\, e^{-\,\bar{\mathbb{A}}^2_{\lambda,t}(\varepsilon)}\right). \end{aligned}$$

By part 2 of Lemma 2 applied to $\gamma = I$, we find that for any positive integer j

$$\frac{d}{d\varepsilon}\mathrm{str}\left(e^{-\bar{\mathbb{A}}^2_{\lambda,t}(\varepsilon)}\right)_{[2j-1]} = \lambda\, d\left[\mathrm{str}\left(\sigma\, e^{-\bar{\mathbb{A}}^2_{\lambda,t}(\varepsilon)}\right)\right]_{[2j-2]} = 0$$

as a consequence of which $\mathrm{str}\left(e^{-\bar{\mathbb{A}}^2_{\lambda,t}(\varepsilon)}\right)_{[2j-1]}$ is actually independent of ε and

$$\mathrm{str}\left(e^{-\bar{\mathbb{A}}^2_{\lambda,t}}\right)_{[2j-1]} = \mathrm{str}\left(e^{-\bar{\mathbb{A}}^2_{t}}\right)_{[2j-1]}.$$

By formula (12) in Theorem 2, the limit on either side therefore exists as $t \to 0$ and

$$\begin{aligned}\lim_{t\to 0}\mathrm{str}\left(e^{-\bar{\mathbb{A}}^2_{\lambda,t}}\right)_{[2j-1]} &= \lim_{t\to 0}\mathrm{ch}\,(\mathbb{A}_t)_{[2j-1]}\\ &= \frac{(-1)^j 2^{j-1}}{(2j-1)!!}\,\mathrm{sres}\left(|\mathbb{A}|^{2j-1}\right)_{[2j-1]}.\end{aligned} \tag{18}$$

2. We now derive a transgression formula for $\mathrm{str}\left(e^{-\bar{\mathbb{A}}^2_{\lambda,t}}\right)_{[2j-1]}$, from which will then follow a transgression formula for $\mathrm{sres}\left(|\mathbb{A}|^{2j-1}\right)_{[2j-1]}$ as a consequence of (18).

$$\begin{aligned}\frac{d}{dt}\mathrm{str}\left(e^{-\bar{\mathbb{A}}^2_{\lambda,t}}\right) &= -\mathrm{str}\left([\bar{\mathbb{A}}_{\lambda,t}, \frac{d}{dt}\bar{\mathbb{A}}_{\lambda,t}]\, e^{-\bar{\mathbb{A}}^2_{\lambda,t}}\right)\\ &= -\mathrm{str}\left([\bar{\mathbb{A}}_{\lambda,t}, \frac{d}{dt}\bar{\mathbb{A}}_{t}\, e^{-\bar{\mathbb{A}}^2_{\lambda,t}}]\right)\\ &= -d\,\mathrm{str}\left(\frac{d}{dt}\bar{\mathbb{A}}_{\lambda,t}\, e^{-\bar{\mathbb{A}}^2_{\lambda,t}}\right).\end{aligned} \tag{19}$$

The first part of Lemma 2 provides a control as $t \to 0$ and as $t \to \infty$ on the asymptotic behaviour of the last expression $\mathrm{str}\left[\frac{d}{ds}\bar{\mathbb{A}}_{\lambda,s}\, e^{-\bar{\mathbb{A}}^2_{\lambda,s}}\right]$ arising in (19). Its primitive in t

$$\tilde{\eta}_\lambda(t) := \int_t^\infty \mathrm{str}\left[\frac{d}{ds}\bar{\mathbb{A}}_{\lambda,s}\, e^{-\bar{\mathbb{A}}^2_{\lambda,s}}\right] ds$$

which exists as a consequence of the invertibility of $D(\lambda)$, has a similar asymptotic behaviour as $t \to 0$. Integrating (19) from t to ∞ and taking the finite part as $t \to 0$ we find that the η invariant (we borrow notations from [L], see his formula (3.19))

$$\tilde{\eta}_\lambda := \mathrm{fp}_{t=0}\tilde{\eta}_\lambda(t) = \mathrm{fp}_{t=0}\int_t^\infty \mathrm{str}\left[\frac{d}{ds}\bar{\mathbb{A}}_{\lambda,s}\, e^{-\bar{\mathbb{A}}^2_{\lambda,s}}\right] ds$$

trangresses $\mathrm{fp}_{t=0}\mathrm{str}\left(e^{-\bar{\mathbb{A}}^2_{\lambda,t}}\right)$:

$$\mathrm{fp}_{t=0}\mathrm{str}\left(e^{-\bar{\mathbb{A}}^2_{\lambda,t}}\right) = d\,\tilde{\eta}_\lambda.$$

But by the first part of the theorem (see equation (18)), this leads to

$$\mathrm{sres}\left(|\mathbb{A}_\lambda|^{2j-1}\right)_{[2j-1]} = \frac{(-1)^j(2j-1)!!}{2^{j-1}} d\,(\tilde{\eta}_\lambda)_{[2j-2]}.$$

8 Relation to hamiltonian gauge anomalies

We first review a finite dimensional situation which will serve as a model for infinite dimensional genralisations. We consider the finite-dimensional Grassmann manifold $\mathrm{Gr}(n,n)$ consisting of rank n projections in $\mathbb{C}^{2n}$, which we parametrise by grading operators $F = 2P - 1$, where P is a finite rank projection.

Lemma 3 *The even forms*

$$\omega_{2j} = \mathrm{tr}\left(F(dF)^{2j}\right), \tag{20}$$

where $j = 1, 2, \ldots$ *are closed forms on* $\mathrm{Gr}(n, n)$.

Proof: By the traciality of tr we have

$$\begin{aligned} d\,\omega_{2j} &= d\,\mathrm{tr}\left(F(dF)^{2j}\right) \\ &= \mathrm{tr}\left((dF)^{2j+1}\right) \\ &= \mathrm{tr}\left(F^2\,(dF)^{2j+1}\right) \\ &\quad \text{since} \quad F^2 = 1 \\ &= -\mathrm{tr}\left(F\,(dF)^{2j+1}\,F\right) \\ &\quad \text{since} \quad F\,dF = -dF\,F \\ &= -\mathrm{tr}\left((dF)^{2j+1}\,F^2\right) \\ &\quad \text{since} \quad \mathrm{tr}([A,B]) = 0 \\ &= -\mathrm{tr}\left((dF)^{2j+1}\right) \\ &= 0. \end{aligned} \tag{21}$$

□

In fact it turns out that the cohomology of $\mathrm{Gr}(n, n)$ is generated by even (nonnormalized) forms of the type $\omega_{2j}, j = 1, \cdots, n$ [MS].

Let us now consider the infinite dimensional geometric setup described in the previous section up to the fact that $\pi : \mathbb{M} = M \times B \to B$ is now a trivial fibration with typical fibre a closed (Riemannian) spin manifold M.
On each open subset $U_\lambda := \{b \in B, \lambda \notin \mathrm{spec}(D_b)\} \subset B$ there is a well defined map [5]

$$\begin{aligned} F : B &\to C\ell_0(M, E) \\ b &\mapsto F_b := (D_b - \lambda I)/|D_b - \lambda I|. \end{aligned}$$

Since $F_b^2 = F_b$, $P_b := \frac{I+F_b}{2}$ is a projection, the range $\mathrm{Gr}(M, E) := \mathrm{Im}F$ of F coincides with the Grassmannian consisting of classical pseudodifferential projections P with kernel and cokernel of infinite rank, acting in the complex Hilbert space $H := L(M, E)$. Here $L^2(M, E)$ denotes the space of square-integrable sections of the vector bundle E over the compact manifold M.
This map $b \mapsto F_b$ is generally not contractible and we want to define cohomology classes on B as in (11) up to some modifications required by the specific situation. This problem usually arises in hamiltonian quantization in field theory, when M is an odd dimensional manifold, the physical space. Although here we deal with even forms for odd order operators, there is a relation to the previous discussion on odd forms for odd order operators which is explained in the end of this section. The problem here is similar in spirit to the earlier discussion in as far as we want to modify the naive cohomology classes, imitating the finite-dimensional case, by local corrections arising from the infinite dimensionality of the problem.
Indeed, in this infinite dimensional setup traces are generally ill-defined, so that we cannot a priori extend the above computation to $\mathrm{Gr}(M, E)$. [6].
However, we can define an analog of (20) at the cost of replacing the trace by a weighted trace.

Proposition 7 *Let* $Q \in C\ell(M, E)$ *be a* **fixed** *admissible elliptic operator with positive order. The exterior differential of the form*

$$\omega_{2j}^Q(F) = \mathrm{tr}^Q\left(F(dF)^{2j}\right) \tag{22}$$

[5]Note here again, the difference in conventions compared to [L]. We follow [CMM], [CM], whereas in [L] the operators D_b are perturbed as $D_b \mapsto D_b + h(D_b)$ by a smoothing function h to avoid the zero modes.

[6]Unless we restrict to the submanifold $\mathrm{Gr}_{res}(M, E) \subset \mathrm{Gr}(M, E)$ consisting of points F such that $F - \epsilon$ is Hilbert-Schmidt for some fixed point $\epsilon \in \mathrm{Gr}(M, E)$ [PS]

on $\mathrm{Gr}(M,E)$*:*

$$d\omega_{2j}^Q = \frac{1}{2q}\mathrm{res}\left([\log Q, F](dF)^{2k+1}F\right).$$

is a local expression which only depends on F *modulo smoothing operators.*

Proof: The locality and the dependence on F modulo smoothing operators follow from the expression of the exterior differential in terms of a Wodzicki residue. To derive this expression, we mimic the finite dimensional proof, taking into account that this time tr^Q is not cyclic:

$$\begin{aligned}
d\omega_{2j}^Q &= d\mathrm{tr}^Q\left(F(dF)^{2j}\right)\\
&= \mathrm{tr}^Q\left((dF)^{2j+1}\right)\\
&= \mathrm{tr}^Q\left(F^2(dF)^{2j+1}\right)\\
&= -\mathrm{tr}^Q\left(F(dF)^{2j+1}F\right)\\
&\quad \text{since}\quad F\,dF = -dF\,F\\
&= \frac{1}{q}\mathrm{res}\left([\log Q, F](dF)^{2j+1}F\right) - \mathrm{tr}^Q\left((dF)^{2j+1}F^2\right)\\
&= \frac{1}{q}\mathrm{res}\left([\log Q, F](dF)^{2j+1}F\right) - \mathrm{tr}^Q\left((dF)^{2j+1}\right),
\end{aligned}$$

where we have used (3) to write

$$\mathrm{tr}^Q\left([F,(dF)^{2j+1}F]\right) = -\frac{1}{q}\mathrm{res}\left(F\left[(dF)^{2j+1}F, \log Q\right]\right) = \frac{1}{q}\mathrm{res}\left([F,\log Q]\,(dF)^{2j+1}F\right).$$

Hence

$$\mathrm{tr}^Q\left(F^2(dF)^{2j+1}\right) = \frac{1}{2q}\mathrm{res}\left([\log Q, F](dF)^{2j+1}F\right)$$

from which the result then follows. $\square$

Let us consider the map

$$\begin{aligned}
\sigma : B &\to C\ell_0(M,E)/Cl_{-\infty}(M,E)\\
b &\mapsto \bar{F}(b) := p\circ F(b)
\end{aligned}$$

where $p : C\ell_0(M,E)\to C\ell_0(M,E)/Cl_{-\infty}(M,E)$ is the canonical projection map. In (quantum field theoretic) applications the map $b\mapsto\sigma(b)=\bar{F}_b$ can be contractible without the map $b\mapsto F_b$ being contractible, a situation which can occur when B is contractible. To justify this, let us first observe that dicontinuities of F give rise to jumps measured by smoothing operators (see e.g. [Me]); indeed, since D_b is a smooth family of self-adjoint elliptic operators on a closed manifold, the discontinuities of F_b are measured by differences of projections $P_{b,\mu}$ over the *finite dimensional* space generated by eigenvectors of $D-\lambda I$ with eigenvalues in $[0,\mu]$ or $[\mu,0]$ according to whether μ is positive or negative. Finite rank projections being smoothing, it follows that the discontinuities are measured by smoothing operators so that the projected map $b\mapsto\bar{F}_b$ turns out to be continuous. Hence if B is contractible, the map σ is contractible.

Example 5 *A standard example in physics is the case when*

- *the base space* B *is a subset of connections in a (hermitian) finite-rank vector bundle* $E\to M$ *over a closed (Riemannian) spin manifold* M*. Since the space of connections in a fixed vector bundle is an affine space, if* B *is the whole space of connections, it is contractible and the map* σ *is indeed contractible.*
- *the operators* D_∇ *(written* D_A *with* $\nabla = d+A$ *in local coordinates) are (twisted) Dirac operators coupled to a connection* $\nabla\in B$*.*

The following theorem builds "renormalised" forms from the original forms ω_{2j}^Q.

Theorem 4 *When $\sigma(B)$ is contractible, there are even forms θ_{2j}^{Q} such that*

$$\omega_{2j}^{ren} = \omega_{2j}^{Q} - \theta_{2j}^{Q}$$

is closed. The forms θ_{2j}^{Q} vanish when the order of $(dF)^{2j+1}$ is less than -dim M. This holds in particular if the order of $(dF)^{2j}$ is less than -dim M in which case $\omega_{2j}^{ren} = \omega_{2j}^{Q} = \mathrm{tr}(F(dF)^{2j})$ is independent of Q.

Remark 10 *In the case of Dirac operators parametrised by gauge connections the order of $(dF)^{2j+1}$ is less than -dim M for all $j > 0$ if the dimension of M is less or equal to -3. This known fact is seen by a simple asymptotic expansion of the differential dF. Using any fixed local trivialization of the underlying vector bundle E, we write $D_\nabla = D + A$ and $F = (D + A)\,|D + A|^{-1}$ where D is the ordinary Dirac operator on $\mathbb{R}^n$ so that the infinitesimal variation dF coincides up to order 1 in dA with $(D + dA)\,|D + dA|^{-1} - D\,|D|^{-1}$. One can check that the square of the operator $|D|^{-1}(1 - \frac{1}{2}(D^{-1}dA + dAD^{-1}))$ is equal to $(D + dA)^{-2}$ up to operators of order -3. Hence $dF = (D + dA)|D + dA|^{-1} = D/|D| - \frac{1}{2}|D|^{-1}D^{-1} \times [dA, D] + \ldots$ up to operators of order -2. Here one has to take into account that the commutator of $|D|$ with A is of order zero. It follows that $D/|D|$ differs from $(D + dA)/|D + dA|$ by an operator of order -1 so that dF has order -1.*
This argument fails for the case of families of metrics because the perturbations of Dirac operators are differential operators of order one. It therefore does not extend to the case of Dirac operators parametrised by metrics since in that case the principal symbol depends on the parameters and the differential dF is a zero order operator.

Proof of Theorem 4: The form $d\omega_{2j}^{Q}$ being a Wodzicki residue, by Prop. 7, it only depends on the projection $\bar{F}$ and is therefore a pull-back by the projection map p of a form β_{2j}^{Q}. The pull-back of β_{2j}^{Q} with respect to σ is a closed form θ_{2j+1}^{Q} on B which is exact since σ is contractible. Indeed, selecting a contraction σ_t with $\sigma_1 = \sigma$ and σ_0 a constant map, we have the standard formula for the potential, $d\theta_{2j}^{Q} = \theta_{2j+1}^{Q}$, with

$$\theta_{2j} = \frac{1}{2j+1}\int_0^1 t^{2j}\iota_{\dot{\sigma}_t}\theta_{2j+1}^{Q}(\sigma_t)dt. \tag{23}$$

where ι_X is the contraction by a vector field X and the dot means differentiation with respect to the parameter t.
When the order of $(dF)^{2j+1}$ is less than -dim M the correction terms θ_{2j}^{Q} vanish and if the order of $(dF)^{2j}$ is less than -dim M, the weighted trace tr^Q coincides with the usual trace so that the naive expression ω_{2j}^{Q} is a closed form independent of Q. $\square$

The 2-form case arises in the quantum field theory gerbe [CMM]. Let B be a contractible parameter space for Dirac operators. For each real number λ, let as before $U_\lambda \subset B$ be the set of parameters for which the Dirac operator $D(\lambda) = D - \lambda I$ does not have λ as an eigenvalue. Denote $U_{\lambda\lambda'} = U_\lambda \cap U_{\lambda'}$ and let $L_{\lambda\lambda'}(A)$ be the top exterior power of the spectral subspace $E_{\lambda\lambda'}$ defined by $\lambda < D_A < \lambda'$ for $A \in U_{\lambda\lambda'}$. The complex lines $L_{\lambda\lambda'}(A)$ form a complex line bundle $L_{\lambda\lambda'}$ over $U_{\lambda\lambda'}$. For $\lambda < \lambda' < \lambda''$ we have the canonical identification

$$L_{\lambda\lambda'} \otimes L_{\lambda'\lambda''} = L_{\lambda\lambda''}. \tag{24}$$

This family of line bundles defines a gerbe over B. Since B is contractible this gerbe is trivial in the sense that

$$L_{\lambda\lambda'} = L_\lambda \otimes L_{\lambda'}^{*} \tag{25}$$

for some line bundles $L_\lambda \to U_\lambda$. The curvature of $L_{\lambda\lambda'}$ is $1/2\pi$ times

$$\omega_2{}^{\lambda\lambda'} = \mathrm{tr}\left(P(\lambda\lambda')(dP(\lambda\lambda'))^2\right) \tag{26}$$

where $P(\lambda\lambda')$ is the projection onto $E_{\lambda\lambda'}$.
Denote as before by $F(\lambda)$ the grading operator $(D_A - \lambda)/|D_A - \lambda|$ on U_λ and let $P(\lambda) = \frac{1}{2}(F(\lambda) + 1)$

be the corresponding spectral projection. In the Hilbert-Schmidt case, when the grading operators are in $\mathrm{Gr}_{\mathrm{res}}(M,E)$ one proves by a direct computation that

$$
{\omega_2}^{\lambda\lambda'} = {\omega_2}^{\lambda} - {\omega_2}^{\lambda'} \tag{27}
$$

on $U_{\lambda\lambda'}$ with

$$
{\omega_2}^{\lambda} = \frac{1}{8}\mathrm{tr}\left(F(\lambda)(dF(\lambda))^2\right) = \mathrm{tr}\left(P(\lambda)(dP(\lambda))^2\right). \tag{28}
$$

Remark 11 *This last equality follows from the cyclicity of the trace on Hilbert-Schmidt operators. Indeed, since* $d\,P(\lambda)\,P(\lambda) + P(\lambda)\,d\,P(\lambda) = d\,P(\lambda)$ *we have*

$$
\begin{aligned}
\mathrm{tr}\left(F(\lambda)(dF(\lambda))^2\right) &= 8\,\mathrm{tr}\left(P(\lambda)(dP(\lambda))^2\right) - 4\,\mathrm{tr}\left((dP(\lambda))^2\right)\\
&= 8\,\mathrm{tr}\left(P(\lambda)(dP(\lambda))^2\right) - 4\,d\mathrm{tr}\left(P(\lambda)\,dP(\lambda)\right)\\
&= 8\,\mathrm{tr}\left(P(\lambda)(dP(\lambda))^2\right) + 4\,d\mathrm{tr}\left(dP(\lambda)\,P(\lambda)\right)\\
&= 8\,\mathrm{tr}\left(P(\lambda)(dP(\lambda))^2\right) - 4\,d\mathrm{tr}\left(dP(\lambda)\,P(\lambda)\right)\\
&= 8\,\mathrm{tr}\left(P(\lambda)(dP(\lambda))^2\right).
\end{aligned}
$$

In the general case the forms ${\omega_2}^{\lambda}$ have to be replaced by the 'renormalised' forms as in Theorem 4. However, we still have

Theorem 5 *The cocycle of forms* ${\omega_2}^{\lambda\lambda'}$ *is trivialized by* $\frac{1}{8}$ *times the forms*

$$
{\omega_2}^{\lambda} = \mathrm{tr}^Q\left(F(\lambda)(dF(\lambda))^2\right) - \theta_2^Q \tag{29}
$$

or equivalently by $\frac{1}{8}$ *times the forms*

$$
{\rho_2}^{\lambda} = 8\,\mathrm{tr}^Q\left(P(\lambda)(dP(\lambda))^2\right) - \theta_2^Q
$$

where θ_2^Q *is as in Theorem 4 (with* $j=1$*), restricted to the open set* U_λ*. In particular,*

$$
\begin{aligned}
{\omega_2}^{\lambda} - {\omega_2}^{\lambda'} &= {\rho_2}^{\lambda} - {\rho_2}^{\lambda'}\\
&= \mathrm{tr}\left(F(\lambda)(dF(\lambda))^2 - F(\lambda')(dF(\lambda'))^2\right)\\
&= \mathrm{tr}\left(P(\lambda)(dP(\lambda))^2 - P(\lambda')(dP(\lambda'))^2\right).
\end{aligned} \tag{30}
$$

Proof: First observe that the correction terms θ_2^Q arising in the differences of the forms ${\omega_2}^{\lambda}$ on the intesections $B_{\lambda\lambda'}$ cancel: they do not depend on the parameter λ since a change of λ gives rise to finite rank perturbations of $F(\lambda)$ and hence to smoothing pertubrations on which the Wodzicki residue vanishes.
Let us show first (30). For $\lambda < \lambda'$ we have

$$
\begin{aligned}
&\mathrm{tr}^Q\left((dP(\lambda))^2\right) - \mathrm{tr}^Q\left((dP(\lambda'))^2\right)\\
=\;& \mathrm{tr}^Q\left((dP(\lambda))^2 - (dP(\lambda'))^2\right)\\
=\;& \mathrm{tr}^Q\left((dP(\lambda\lambda'))^2 + dP(\lambda')dP(\lambda\lambda') + dP(\lambda\lambda')dP(\lambda')\right)\\
=\;& \mathrm{tr}\left(dP(\lambda')dP(\lambda\lambda') + dP(\lambda\lambda')dP(\lambda')\right)\\
=\;& 0,
\end{aligned}
$$

since $\mathrm{tr}(dP)^2 = 0$ for any finite rank projector and by cyclicity of the ordinary trace, from which it follows that

$$
\mathrm{tr}^Q\left(F(\lambda)(dF(\lambda))^2\right) - \mathrm{tr}^Q\left(F(\lambda')(dF(\lambda'))^2\right) = 8\left(\mathrm{tr}^Q\left(P(\lambda)(dP(\lambda))^2\right) - \mathrm{tr}^Q\left(P(\lambda')(dP(\lambda'))^2\right)\right).
$$

To show (29) we expand ω_2^{λ} in powers of the difference projection $P(\lambda\lambda')$ and observe that the zeroth order term is equal to $\omega_2^{\lambda'}$, the third order term is $\omega_2^{\lambda\lambda'}$. The mixed terms are ordinary traces, since

the operators contain the finite rank projector $P(\lambda\lambda')$ as a factor; using the cyclicity of the trace and repeatedly $dPP = PdP = dP$ for any projector and $dPP' + PdP' = 0$ for any pair of mutually orthogonal projectors, we get

$$\begin{aligned} & \omega_2^{\lambda} - \omega_2^{\lambda'} \\ = \quad & \mathrm{tr}\,(P(\lambda\lambda')dP(\lambda\lambda')dP_{\lambda\lambda'}) + \mathrm{tr}\,(P(\lambda')dP(\lambda')dP(\lambda\lambda')) + \\ + \quad & \mathrm{tr}\,(P(\lambda')dP(\lambda\lambda')dP(\lambda')) + tr\,(P(\lambda')dP(\lambda\lambda')dP(\lambda\lambda')) \\ + \quad & \mathrm{tr}\,(P(\lambda\lambda')dP(\lambda')dP(\lambda')) + \mathrm{tr}\,(P(\lambda\lambda')dP(\lambda')dP(\lambda\lambda')) + \mathrm{tr}\,(P(\lambda\lambda')dP(\lambda\lambda')dP(\lambda')) \\ = \quad & \omega_2^{\lambda\lambda'} - 3\mathrm{tr}\,((1 - P(\lambda') - P(\lambda\lambda'))dP(\lambda')dP(\lambda\lambda'))\,. \end{aligned}$$

Next for any triple of mutually orthogonal projectors one has $\mathrm{tr}\,(PdP'dP'') = 0$, again by the above mentioned operator identities; applying this to $P = 1 - P(\lambda') - P(\lambda\lambda')$, $P' = P(\lambda')$ and $P'' = P(\lambda\lambda')$ we see that the mixed terms on the right-hand-side of the above equation vanish and the Theorem follows. $\square$

Remark 12 *Actually, this is just the degree 2 cohomology part of the statement that the Chern characters for direct summand in vector bundles add up to the Chern character of the sum; for the chosen curvature forms the statement is valid on the level of de Rham forms, not just for de Rham classes.*

The forms ω_2^{λ} are related but not equal to the gerbe eta forms studied in [L]. Let $\mathcal{G}$ be the group of smooth based group gauge transformations acting on smooth sections $C^{\infty}(M, E)$ as unitary operators. On the base $B = \mathcal{A}/\mathcal{G}$ equipped with the open cover $V_{\lambda} = \pi(U_{\lambda})$ we have well-defined eta forms η_{λ} such that $d\eta_{\lambda} = \Omega_3$, where Ω_3 is the Dixmier-Douady 3-cohomology class classifying the gerbe on the base B. (Here we ignore possible torsion in cohomology and work with de Rham representatives.) Let $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{G}$ be the canonical projection where $\mathcal{G}$ is the gauge group. The pull-back $\pi^*(\Omega_3)$ is exact, $d\Theta = \pi^*(\Omega_3)$. These are related to ω_2^{λ}, as cohomology classes, by

$$[\omega_2^{\lambda}] = [\Theta - \pi^*(\eta_{\lambda})].$$

The above relation holds only in cohomology, not as a relation of forms. This is related to the fact that the difference of the pull-back forms $\pi^*(\eta_{\lambda})$ must vanish in gauge directions whereas the difference $\omega_2^{\lambda} - \omega_2^{\lambda'} = \omega_2^{\lambda\lambda'}$ is nonvanishing even in gauge directions. However, we have

Proposition 8 *Restricted to gauge orbits, the forms $\omega_2^{\lambda\lambda'}$ vanish as cohomology classes. More precisely, $\omega_2^{\lambda\lambda'} = d\omega_1^{\lambda\lambda'}$ on gauge orbits, where*

$$\omega_1^{\lambda\lambda'}(X) = -\mathrm{tr}\,(\,XP(\lambda\lambda'))\,.$$

Here we can use ordinary trace since $P(\lambda\lambda')$ has finite rank; X is an element of the Lie algebra of $\mathcal{G}$ acting as multiplication operator in the Hilbert space H.

Proof: The gauge group acts on projections by conjugation $P \mapsto gPg^{-1}$ and so

$$\begin{aligned} & (d\omega_1^{\lambda\lambda'})(X, Y) \\ = \quad & \mathcal{L}_X\omega_1^{\lambda\lambda'}(Y) - \mathcal{L}_Y\omega_1^{\lambda\lambda'}(X) - \omega_1^{\lambda\lambda'}([X, Y]) \\ = \quad & -\mathrm{tr}\,(Y[P(\lambda\lambda'), X]) + \mathrm{tr}\,(X[P(\lambda\lambda'), Y]) + \mathrm{tr}\,([X, Y]P(\lambda\lambda')) \\ = \quad & -\mathrm{tr}\,([X, Y]P(\lambda\lambda')) \\ = \quad & \mathrm{tr}\,(P(\lambda\lambda')[[P(\lambda\lambda'), X], [P(\lambda\lambda'), Y]]) \\ = \quad & \omega_2^{\lambda\lambda'}(X, Y), \end{aligned}$$

where in the last step we have used the projection property $P^2 = P$ and the cyclicity of trace for finite rank operators; $\mathcal{L}_X$ is the Lie derivative by vector field along gauge orbits corresponding to the conjugation action of the group $\mathcal{G}$. $\square$

Remark 13 A similar modification can be made for the gauge action on the local forms ω_2^λ to show that the action is consistent on overlaps. Again, restricting to gauge orbits, using $F^2 = 1$ and rearranging terms, one can write

$$\begin{aligned}
& \mathrm{tr}^Q\left(F(\lambda)[[F(\lambda), X], [F(\lambda), Y]\right) \\
= \quad & -4\mathrm{tr}^Q\left([X,Y]F(\lambda)\right) + 2\mathrm{tr}^Q\left(2[XF(\lambda), Y] + [F(\lambda)XF(\lambda)Y, F(\lambda)] + [XY, F(\lambda)]\right) \\
= \quad & -4\mathrm{tr}^Q\left([X,Y]F(\lambda)\right) + 2\mathrm{res}\left(2[\log Q, XF(\lambda)]Y + [\log Q, XY]F(\lambda) + [\log Q, F(\lambda)XF(\lambda)Y]F(\lambda)\right)
\end{aligned}$$

where on the right the first term is a trivial cocycle and the rest, being a residue, does not depend on finite rank perturbations and in particular not on the parameter λ.

Here we have only discussed the cocycles of degree 2 because they are the most relevant in gauge theory; it is clear that similar computations can be performed with the higher cocycles.
The 2-forms ω_2^λ are directly 'seen' in quantum field theory in the following way. These forms appear as curvature forms of local vacuum line bundles for fermion field in gauge background, [CMM, EM]. The gauge action on gauge connections lifts to an action of an extension of the group of gauge transformations on the local line bundles. On the Lie algebra level, the 2-cocycle describing the Lie algebra extension("hamiltonian anomaly" [Mi]) is just the curvature form evaluated in the gauge directions. In the case of fields in one space dimension, this extension (for a simple compact gauge group) defines an affine Kac-Moody algebra.

References

[B] J.-M. Bismut, *The Atiyah-Singer index theorem for families of Dirac operators: two equation proofs*, Invent. Math. **83** (1986) 91-151

[BC] J -M. Bismut, J. Cheeger, *η-invariants and their adiabatic limits*, Journ . Amer. Math. Soc. **2** (1989) 33-70

[BF] J.-M. Bismut, D. Freed, *The analysis of elliptic families I and II*, Commun. Math. Phys. **106** (1986) 159-176 and Comm. Math. Phys. **107** (1986) 103-163

[BGV] N. Berline, E. Getzler, M. Vergne, *Heat kernels and Dirac operators. Grundlehren der Mathematischen Wissenschaften* **298**, Springer Verlag, Berlin 1992.

[CM] A. L. Carey and J. Mickelsson: *The universal gerbe, Dixmier-Douady class, and gauge theory* Lett. Math. Phys. **59** (2002) 47-60

[CMM] A. L. Carey, J. Mickelsson and M. Murray: *Index Theory, Gerbes, and Hamiltonian Quantization* Commun. Math. Phys. 183 (1997) 707-722

[CoM] A. Connes, H. Moscovici, *The local index formula in non commutative geometry*, Geom. Funct. Anal. **5** (2) (1995) 174– 243

[EM] C. Ekstrand, J. Mickelsson, *Gravitational anomalies, gerbes, and Hamiltonian quantization*

[H] N. Higson, *The local index formula in noncommutative geometry*, Trieste Lecture Notes (2004); *The residue index theorem of Connes and Moscovici*, Clay Mathematics Proceedings (2004). (http://www.math.psu.edu/higson/ResearchPapers.html) Commun. Math. Phys. **212** (2000) 613-624

[KV] M. Kontsevich, S. Vishik, *Geometry of determinants of elliptic operators*, Func. Anal. on the Eve of the XXI century, Vol I, Progress in Mathematics **131** (1994) 173-197 ; *Determinants of elliptic pseudodifferential operators*, Max Planck Preprint (1994)

[LP] J.-M. Lescure, S. Paycha, *Traces on pseudodifferential operators and associated determinants* Preprint (2005)

[L] J. Lott, *Higher degree analogs of the determinant bundle* Commun. Math. Phys. **230** (2002) 41-69

[Me] R. Melrose, *The Atiyah-Patodi-Singer index theorem*, Res. Notes in Math.**4**, (1993)

[Mi] J. Mickelsson, *Current algebras and groups*, Plenum Press, London and New York (1989)

[MS] J. W. Milnor and J. D. Stasheff, *Characteristic classes*, Annals of Mathematics Studies, No. 76. Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo (1974)

[O] K. Okikiolu, *The Campbell-Hausdorff theorem for elliptic operators and a related trace formula*, Duke. Math. Journ. **79** (1995) 687–722

[PR] S. Paycha, S. Rosenberg, *Curvature on determinant bundles and first Chern forms*, Journ. Geom. Phys. 45 (2003) 393-429 ; *Traces and characteristic classes in loop groups*, in *Infinite dimensional Groups and Manifolds*, Ed. T. Wurzbacher, I.R.M.A. Lectures in Mathematical and Theoretical Physics **5**, de Gruyter (2004) 185–212

[PS] A. Pressly, G. Segal, *Loop groups*, Oxford Monographs, Oxford Science, Clarendon Press Publications (1986)

[PS1] S. Paycha, S. Scott *Chern-Weil forms associated with superconnections* in "Analysis, geometry and topology of elliptic operators", Ed. B. Booss-Bavnbeck, S. Klimek, M. Lesch, W. Zhang, World Scientific (2006)

[PS2] S. Paycha, S. Scott, *A Laurent expansion for regularised integrals of holomorphic symbols*, to appear in Geom. Funct. Anal.

[Q] D. Quillen, *Superconnections and the Chern character*, Topology **24** (1985) 89–95

[Sc] S. Scott, *Zeta-Chern forms and the local family index theorem*, Trans. Amer. Math. Soc., to appear. arXiv: math.DG/0406294.

[W] M. Wodzicki, *Non commutative residue* in Lecture Notes in Math. **1283**, Springer Verlag 1987; *Spectral asymmetry and noncommutative residue* (in Russian), Thesis, Steklov Institute (former) Soviet Academy of Sciences, Moscow, 1984